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Algebraic cycles and homotopy theory

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1. Introduction and statement of results

One of the remarkable features of an algebraic variety, when considered as a geometric object, is its wealth of subvarieties. In fact the family of subvarieties itself forms an algebraic space, and over the years much serious thought has been devoted to understanding its structure. In general it is quite complicated. However, we shall show that the key to deciphering its global topology is to examine its homotopy groups which, after a certain idealization of the space, turn out to be astonishingly simple. In fact for complex projective space $\mathbf{P}^n$ the structure can be understood completely. This yields new information about the topology of the classical Chow varieties, and establishes an explicit relationship with universal cohomology operations. In the general case complete computations are difficult, but we shall establish a "complex suspension" theorem and lay the foundations for a theory based on the homotopy groups of Chow varieties.

To state the results we must give precise meaning to "the space of subvarieties". We begin with the fundamental case of subvarieties in complex projective n-space $\mathbf{P}^n$. For each pair of integers p and d with $d \geq 1$ and $0 \leq p < n$, consider the set $\mathscr{C}_{p,d}(\mathbf{P}^n)$ of all finite formal sums

$$c = \sum n_\alpha V_\alpha$$

where for each α, n_α is a positive integer and $V_\alpha \subset \mathbf{P}^n$ is an irreducible algebraic subvariety of dimension p, and where

$$\deg(c) \stackrel{\text{def}}{=} \sum n_\alpha \deg(V_\alpha) = d$$

(i.e., $[c] = d[\mathbf{P}^p]$ in $H_{2p}(\mathbf{P}^n; \mathbf{Z})$). Each space $\mathscr{C}_{p,d}(\mathbf{P}^n)$ can be realized canonically as a projective algebraic variety, and is called a "Chow variety" (cf. [S], [Sh]). In particular, it has the structure of a compact Hausdorff space. As we shall see, this topology agrees with most of the other natural candidates. For

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example, it agrees with the one induced by the embedding of $\mathscr{C}_{p,d}(\mathbf{P}^n)$ into the dual space of the space of differential $2p$-forms on $\mathbf{P}^n$. It also agrees with the flat-norm topology coming from geometric measure theory.

As an algebraic variety $\mathscr{C}_{p,d}(\mathbf{P}^n)$ can be quite complicated. In general it is singular and in fact reducible with many algebraic components of varying dimensions. Nevertheless, elementary general position arguments (see §2) show it to be connected and simply-connected. The main point of this paper is to show that, in fact, the entire homotopy structure of $\mathscr{C}_{p,d}(\mathbf{P}^n)$ stabilizes to a simple, computable one as $d \to \infty$. This fact is captured by passing to a limit as follows.

Fix a distinguished, p-dimensional linear subspace $\ell_0 \subset \mathbf{P}^n$ and for each degree $d \geq 1$ consider the topological (in fact analytic) embedding

$$\mathscr{C}_{p,d}(\mathbf{P}^n) \subset \mathscr{C}_{p,d+1}(\mathbf{P}^n)$$

given by

$$c \to c + \ell_0.$$

Using this sequence of embeddings we can take the union

$$\mathscr{C}_p(\mathbf{P}^n) = \bigcup_{d=1}^{\infty} \mathscr{C}_{p,d}(\mathbf{P}^n)$$

and give it the weak topology—where a set F is closed if and only if $F \cap \mathscr{C}_{p,d}(\mathbf{P}^n)$ is closed for all d. This makes $\mathscr{C}_p(\mathbf{P}^n)$ a connected Hausdorff space with the property that if $K \subset \mathscr{C}_p(\mathbf{P}^n)$ is compact, then $K \subset \mathscr{C}_{p,d}(\mathbf{P}^n)$ for some d. Addition of cycles makes $\mathscr{C}_p(\mathbf{P}^n)$ an abelian topological semigroup with unit $[\ell_0]$. This addition respects the degree-filtration and is analytic at each finite level. Notice that $\mathscr{C}_p(\mathbf{P}^n)$ can be thought of as the space of all positive algebraic cycles of dimension p in $\mathbf{P}^n$. It is sometimes useful to adopt the notation

$$\mathscr{C}^q(\mathbf{P}^n) \stackrel{\text{def}}{=} \mathscr{C}_{n-q}(\mathbf{P}^n)$$

where $q = n - p$ is the codimension of the cycles. Our first main result is the following.

THEOREM 1. *For each $q \leq n$ there is a homotopy equivalence*

$$\mathscr{C}^q(\mathbf{P}^n) \cong K(\mathbf{Z},2) \times K(\mathbf{Z},4) \times \cdots \times K(\mathbf{Z},2q)$$

where $K(\mathbf{Z},2k)$ denotes the standard Eilenberg-MacLane space.

Recall that for a finitely generated abelian group G, $K(G,k)$ is the connected, countable CW-complex uniquely determined up to homotopy type by the requirement that $\pi_k(K(G,k)) = G$ and $\pi_j(K(G,k)) = 0$ for $j \neq k$. The space (G,k) classifies the functor $H^k(\,\cdot\,;G)$, and the cohomology of $K(G,k)$

with coefficients in Λ corresponds thereby to cohomology operations of type (G, k, Λ). (See [W].) Therefore, Theorem 1 implies that the cohomology of the cycle space $\mathscr{C}^q(\mathbf{P}^n)$ is given exactly by certain universal cohomology operations. For $\Lambda = \mathbf{Z}$ or $\mathbf{Z}/k\mathbf{Z}$, the cohomology groups $H^*(K(\mathbf{Z}, k); \Lambda)$ have been well understood for some time.

There are two specific cases of Theorem 1 which are illuminating.

Example ($q = 1$). This is the case of divisors. The space $\mathscr{C}_{n-1,d}(\mathbf{P}^n)$ is the projectivization of the space of homogeneous polynomials of degree d in $\mathbf{C}^{n+1}$, i.e., $\mathscr{C}_{n-1,d}(\mathbf{P}^n) = \mathbf{P}^{N(d,n)}$ where $N(d, n) + 1 = \binom{n+d}{d}$. Each inclusion $\mathscr{C}_{n-1,d}(\mathbf{P}^n) \subset \mathscr{C}_{n-1,d+1}(\mathbf{P}^n)$ is a linear embedding $\mathbf{P}^{N(d,n)} \subset \mathbf{P}^{N(d+1,n)}$ and we conclude directly that

$$\mathscr{C}^1(\mathbf{P}^n) = \mathbf{P}^\infty \cong K(\mathbf{Z}, 2).$$

Example ($q = n$). This is the case of cycles of dimension zero. ℓ_0 is a distinguished point. The space $\mathscr{C}_{0,d}(\mathbf{P}^n)$ is just the d-fold symmetric product $\mathrm{SP}^d(\mathbf{P}^n)$ of $\mathbf{P}^n$ and the limit

$$\mathscr{C}^n(\mathbf{P}^n) = \mathrm{SP}(\mathbf{P}^n)$$

is exactly the *infinite symmetric product* as defined by Dold and Thom who proved the following beautiful result.

THE DOLD-THOM THEOREM ([DT, 1-2]). *For any connected finite complex A there is a homotopy equivalence*

$$\mathrm{SP}(A) \cong \prod_{k>0} K(H_k(A; \mathbf{Z}), k).$$

In particular, there is a natural isomorphism

$$\pi_*(\mathrm{SP}(A)) \cong H_*(A; \mathbf{Z}).$$

Setting $A = \mathbf{P}^n$ yields Theorem 1 in the special case where $q = n$. In fact all the theorems in this paper constitute a generalization of the work of Dold and Thom to the context of algebraic cycles.

A brief philosophical digression. In general the algebraic variety $\mathscr{C}_{p,d}(\mathbf{P}^n)$ is highly singular, and there is often merit in normalizing and even resolving these singularities. There are advantages in passing to the reduced Hilbert scheme for example. Here, however, it is important that we work with Chow varieties. We specifically think of the algebraic cycles as a distinguished subspace of all cycles, in the same sense that holomorphic maps from X to Y form a distinguished subspace of all maps—or that self-dual Yang-Mills connections form a distinguished subspace of all connections. Interestingly, these subspaces often have the property that, as degree increases, they form better and better approximations to the space itself. This also happens in our case.

THEOREM 2. *For all n, p and d, the inclusion i: $\mathscr{C}_{p,d}(\mathbf{P}^n) \hookrightarrow \mathscr{C}_p(\mathbf{P}^n)$ has a right homotopy inverse through dimension $2d$.*

This means that there exists a finite complex C and a map j: $C \to \mathscr{C}_{p,d}(\mathbf{P}^n)$ so that the composition $i \circ j$: $C \to \mathscr{C}_p(\mathbf{P}^n)$ is 2d-connected. This implies that the maps induced by i are surjective on homotopy and homology groups, and injective on cohomology in all dimensions $\leq 2d$. In particular, as d becomes large the cohomology of $\mathscr{C}_{p,d}(\mathbf{P}^n)$ becomes quite complicated. At any prime ℓ it carries a significant piece of the Steenrod algebra.

It is reasonable to ask whether the map i: $\mathscr{C}_{p,d}(\mathbf{P}^n) \to \mathscr{C}_p(\mathbf{P}^n)$ is actually 2d-connected. This question remains open at the moment.

Theorem 1 does however assert the following. Let $\mathscr{Z}_{2p}(\mathbf{P}^n)$ denote the space of integral $2p$-cycles of degree zero on $\mathbf{P}^n$ with the flat-norm topology. (See Federer [F].) Then the inclusion

$$\mathscr{C}_p(\mathbf{P}^n) \hookrightarrow \mathscr{Z}_{2p}(\mathbf{P}^n),$$

given on cycles of degree d by mapping $c \to c - d\ell_0$, is a homotopy equivalence. This follows from the theorem of F. Almgren [A] which generalizes the work of Dold and Thom to the context of topological cycles.

Theorem 1 leads to a number of fascinating questions. Note that at the degree-1 level we obtain the Grassmannian

$$\mathscr{C}_{p,1}(\mathbf{P}^{p+q}) = \mathscr{G}_p^q \overset{\text{def}}{=} \frac{U_{p+q+1}}{U_{p+1} \times U_q}$$

of projective p-planes in $\mathbf{P}^{p+q}$. By fixing q and letting p go to infinity, the natural inclusion $\mathscr{C}_{p,1}(\mathbf{P}^{p+q}) \subset \mathscr{C}_p(\mathbf{P}^{p+q})$ gives a map

$$\mathscr{G}_\infty^q \subset \mathscr{C}^q(\mathbf{P}^\infty)$$

which can be reinterpreted as a map

$$BU_q \longrightarrow K(\mathbf{Z},2) \times K(\mathbf{Z},4) \times \cdots \times K(\mathbf{Z},2q).$$

This map turns out to be the total Chern class $(c_1, c_2, \ldots, c_q)$ of the universal q-plane bundle, where we identify $H^{2k}(BU_q; \mathbf{Z})$ with $[BU_q, K(\mathbf{Z}, 2k)]$. Taking a limit with respect to q gives a map

$$BU \longrightarrow \prod_{k>1} K(\mathbf{Z}, 2k)$$

whose injectivity on homotopy groups implies the Bott periodicity theorem. On the homotopy groups π_{2k} the induced map is exactly multiplication by $(k-1)!$.

This together with many related matters is discussed in a paper with M. L. Michelsohn [LM].

Theorem 1 can be generalized to any fixed subvariety $X \subset \mathbf{P}^n$. In fact there are several generalizations depending on how one defines the space $\mathscr{C}_p(X)$. The statements and proofs go through uniformly for a variety of choices. Originally the author chose a very restrictive definition of $\mathscr{C}_p(X)$ which assumed $\ell_0 \subset X$ and then inverted ℓ_0 as above. The definitions introduced below are all due to Eric Friedlander. Each begins with the same naive object, namely the set

$$\mathscr{C}_{p,\cdot}(X)$$

of all positive algebraic p-cycles contained in X, together with a disjoint point 0 representing the "empty" cycle. One then passes to some form of "group completion". Notice that $\mathscr{C}_{p,\cdot}(X)$ forms a closed submonoid of the abelian topological monoid $\mathscr{C}_{p,\cdot}(\mathbf{P}^n) = \coprod_{d \geq 0} \mathscr{C}_{p,d}(\mathbf{P}^n)$. (A *monoid* is a semigroup with unit.) It is useful to break $\mathscr{C}_{p,\cdot}(X)$ into connected components:

$$\mathscr{C}_{p,\cdot}(X) = \coprod_{\alpha \in A} \mathscr{C}_{p,\alpha}(X)$$

where $A = \pi_0(\mathscr{C}_{p,\cdot}(X))$. Each component $\mathscr{C}_{p,\alpha}(X)$ is an algebraic variety, and translation of the whole space by any element is an algebraic map. Suppose now that the monoid A is finitely generated and free. We can then choose a set of cycles $c_1, \dots, c_m$ representing free generators for A and define

$$\mathscr{C}_p(X) = \varinjlim_{\alpha} \mathscr{C}_{p,\alpha}(X)$$

over the directed system of embeddings

$$\mathscr{C}_{p,\alpha}(X) \xhookrightarrow{+c} \mathscr{C}_{p,\alpha+[c]}(X)$$

for $c \in \mathbf{Z}^+ c_1 \oplus \cdots \oplus \mathbf{Z}^+ c_m \cong A$. In the general case we must choose a cycle $c_\alpha \in \alpha$ for each $\alpha \in A$ and consider A as an indexing category for $\mathscr{C}_{p,\cdot}(X)$ with $\operatorname{Hom}(\mathscr{C}_{p,\alpha}(X), \mathscr{C}_{p,\beta}(X)) \cong \{\gamma \in A : \alpha + \gamma = \beta\}$. The relevant diagrams commute up to homotopy (since $c_\alpha + c_\beta$ is connected to $c_{\alpha+\beta}$), and following [Fr3] we can form a homotopy direct limit called the *Friedlander completion*

$$\mathscr{C}_p(X) = \underset{\alpha}{\underrightarrow{\operatorname{Flim}}}\, \mathscr{C}_{p,\alpha}(X)$$

by constructing a mapping telescope for the family of translations $\tau_\alpha(\cdot) = (\cdot) + c_\alpha$. (See §2.)

Note. Eric Friedlander has also suggested defining $\mathscr{C}_p(X) = \Omega B \mathscr{C}_{p,\cdot}(X)$ where $B(M)$ denotes the classifying space of a monoid M given via the classical bar construction. He has shown [Fr3] that the two definitions of $\mathscr{C}_p(X)$ are homotopy equivalent and that the second has certain advantages.

Another important object to consider is the naive group completion $\tilde{\mathscr{C}}_p(X)$ of $\mathscr{C}_{p,\cdot}(X)$. This is simply the free abelian group generated by the irreducible p-dimensional subvarieties of X. We furnish $\tilde{\mathscr{C}}_p(X)$ with the weak topology for the family of compact subspaces $F_{\alpha\beta} = \delta(\mathscr{C}_{p,\alpha}(X) \times \mathscr{C}_{p,\beta}(X))$ where $\delta\colon \mathscr{C}_{p,\cdot}(X) \times \mathscr{C}_{p,\cdot}(X) \to \tilde{\mathscr{C}}_p(X)$ is the quotient map $\delta(c, c') = c - c'$, and where each $F_{\alpha\beta}$ carries the obvious quotient topology. This makes $\tilde{\mathscr{C}}_p(X)$ a topological group. It is the universal topological group associated to the monoid $\mathscr{C}_{p,\cdot}(X)$.

Remark. Results in [DT2] combined with Theorem 1 show that $\tilde{\mathscr{C}}_p(\mathbf{P}^n)$ is homotopy equivalent to $\mathscr{C}_p(\mathbf{P}^n)$ for all p and n. In fact $\tilde{\mathscr{C}}_p(X) \cong \mathscr{C}_p(X)$ for all subvarieties $X \subset \mathbf{P}^n$. Details of this will appear in a paper with Friedlander.

Consider now a linear embedding $\mathbf{P}^n \subset \mathbf{P}^{n+m}$ for some $m \geq 1$ and choose a linear subspace $\mathbf{P}^{m-1} \subset \mathbf{P}^{n+m}$ disjoint from $\mathbf{P}^n$. (All choices are equivalent.) Linear projection away from $\mathbf{P}^{m-1}$ and onto $\mathbf{P}^n$ gives us a holomorphic vector bundle

$$\pi\colon (\mathbf{P}^{n+m} - \mathbf{P}^{m-1}) \longrightarrow \mathbf{P}^n$$

of rank m (m copies of $\mathscr{O}(1)$).

Definition. For any closed subset $A \subset \mathbf{P}^n$ we define the *complex m-fold suspension* of A to be the subset $\Sigma\!\!\!/^m A \subset \mathbf{P}^{n+m}$ given by

$$\Sigma\!\!\!/^m A = \overline{\pi^{-1}(A)}$$

where closure is taken in $\mathbf{P}^{n+m}$.

Note that $\Sigma\!\!\!/^m(A)$ is simply the union of all projective lines joining A to $\mathbf{P}^{m-1}$; i.e., it is the *complex join* of A with $\mathbf{P}^{m-1}$. When $m = 1$, $\Sigma\!\!\!/(A)$ is just the Thom space of $\mathscr{O}(1)|_A$, and in general $\Sigma\!\!\!/^m(A) = \Sigma\!\!\!/(\Sigma\!\!\!/(\dots(\Sigma\!\!\!/(A))\dots))$. In fact $\Sigma\!\!\!/(A)$ is defined in homogeneous coordinates by the same polynomial equations that define A (considered now to have m "secret" variables).

Since the map $\Sigma\!\!\!/^m$ carries subvarieties to subvarieties, it extends naturally to a monoid homomorphism

$$\Sigma\!\!\!/^m\colon \mathscr{C}_{p,\cdot}(X) \longrightarrow \mathscr{C}_{p+m,\cdot}(\Sigma\!\!\!/^m(X)) \tag{1.1}$$

for any subvariety $X \subset \mathbf{P}^n$. We shall prove that this map always induces a bijection on π_0. Thus if $\{c_\alpha\}_{\alpha\in A}$ is the distinguished family of algebraic cycles on X with which we are constructing our homotopy limit $\mathscr{C}_p(X)$, then we can choose the family $\{\Sigma\!\!\!\!/^m c_\alpha\}_{\alpha\in A}$ to construct the analogous limit $\mathscr{C}_{p+m}(\Sigma\!\!\!\!/^m X)$ and the map $\Sigma\!\!\!\!/^m$ extends naturally to these spaces. Our central result is the following:

THEOREM 3 (THE COMPLEX SUSPENSION THEOREM). *Let $X \subset \mathbf{P}^n$ be any algebraic subvariety. Then for every dimension p and every positive integer m, the map*

$$\Sigma\!\!\!\!/^m : \mathscr{C}_p(X) \longrightarrow \mathscr{C}_{p+m}(\Sigma\!\!\!\!/^m X)$$

is a homotopy equivalence. So also is the continuous group homomorphism

$$\tilde{\Sigma\!\!\!\!/}^m : \tilde{\mathscr{C}}_p(X) \longrightarrow \tilde{\mathscr{C}}_{p+m}(\Sigma\!\!\!\!/^m X)$$

induced from (1.1) *by universality.*

Note that by setting $X = \mathbf{P}^n$, $p = 0$, and applying the Dold-Thom theorem, we recover Theorem 1 from the first part of Theorem 3. Setting $p = 0$ and using Dold-Thom also gives the following:

COROLLARY 4. *For any connected projective variety X, there is a natural isomorphism*

$$\pi_*(\mathscr{C}_m(\Sigma\!\!\!\!/^m X)) \cong H_*(X;\mathbf{Z})$$

for each $m \geq 0$.

One wonders whether the Grothendieck or Hodge filtrations on $H^*(X;\mathbf{C})$ can be recovered from Corollary 4. Some progress has recently been made on this in [FrM].

The complex suspension theorem and its corollary make it plausible that the groups $\pi_*\mathscr{C}_*(\cdot)$ constitute an interesting set of invariants for algebraic varieties. They can actually be expanded into a "theory" which includes relative groups, long exact sequences, variable coefficient modules, etc. Our final results will be concerned with such things. We begin with the case of finite coefficients. Fix a projective variety X and a dimension $p \geq 0$. Let k be a positive integer and consider the closed subgroup $k\tilde{\mathscr{C}}_p(X) = \{kc : c \in \tilde{\mathscr{C}}_p(X)\}$. The quotient group $\tilde{\mathscr{C}}_p(X) \otimes \mathbf{Z}_k \overset{\text{def}}{=} \tilde{\mathscr{C}}_p(X)//k\tilde{\mathscr{C}}_p(X)$ with the quotient topology, will be called the group of *algebraic p-cycles* mod k. Algebraically this is merely the free $\mathbf{Z}/k\mathbf{Z}$-module generated by the irreducible p-dimensional subvarieties of X. The topology on this group coincides with Federer's flat-norm topology induced by embedding $\tilde{\mathscr{C}}_p(X) \otimes \mathbf{Z}_k$ into the space of rectifiable $2p$-currents mod k. Note the short exact sequence

$$0 \longrightarrow \tilde{\mathscr{C}}_p(X) \xrightarrow{k} \tilde{\mathscr{C}}_p(X) \longrightarrow \tilde{\mathscr{C}}_p(X) \otimes \mathbf{Z}_k \longrightarrow 0. \tag{1.2}$$

Our main result is that this sequence is a principal fibration and therefore the complex suspension theorem holds for these groups.

THEOREM 5. *Let $X \subset \mathbf{P}^n$ be any algebraic subvariety. Then for all p, m and k the group homomorphism*

$$\Sigma^m : \tilde{\mathscr{C}}_p(X) \otimes \mathbf{Z}_k \longrightarrow \tilde{\mathscr{C}}_{p+m}(\Sigma^m X) \otimes \mathbf{Z}_k$$

induced by complex suspension is a homotopy equivalence.

Applying the Dold-Thom Theorem gives the following:

COROLLARY 6. *For any connected projective variety X there is a natural isomorphism*

$$\pi_*\big(\tilde{\mathscr{C}}_m(\Sigma^m X) \otimes \mathbf{Z}_k\big) \cong H_*(X; \mathbf{Z}_k)$$

for each $m \geq 0$ and $k \geq 1$.

COROLLARY 7. *For each $q \leq n$ and each $k \geq 1$ there is a homotopy equivalence*

$$\tilde{\mathscr{C}}^q(\mathbf{P}^n) \otimes \mathbf{Z}_k \cong K(\mathbf{Z}_k, 2) \times K(\mathbf{Z}_k, 4) \times \cdots \times K(\mathbf{Z}_k, 2q).$$

Note that since (1.2) is a fibration there is an associated long exact sequence of homotopy groups. When $p = 0$ this corresponds exactly—via the Dold-Thom Theorem—to the long homology sequence coming from the short exact coefficient sequence: $0 \to \mathbf{Z} \xrightarrow{k} \mathbf{Z} \to \mathbf{Z}_k \to 0$.

We now take up the "relative" case. Fix projective varieties $Y \subset X \subset \mathbf{P}^n$ and a dimension $p \geq 0$, and consider the closed subgroup $\tilde{\mathscr{C}}_p(Y) \subset \tilde{\mathscr{C}}_p(X)$. The quotient group $\tilde{\mathscr{C}}_p(X, Y) \overset{\text{def}}{=} \tilde{\mathscr{C}}_p(X)//\tilde{\mathscr{C}}_p(Y)$ with the quotient topology, will be called the group of *algebraic p-cycles* on X modulo Y. As before the main point is that the short exact sequence

$$0 \longrightarrow \tilde{\mathscr{C}}_p(Y) \longrightarrow \tilde{\mathscr{C}}_p(X) \longrightarrow \tilde{\mathscr{C}}_p(X, Y) \longrightarrow 0 \tag{1.3}$$

is a principal fibration. This implies:

THEOREM 8. *Let $Y \subset X \subset \mathbf{P}^n$ be any pair of algebraic subvarieties. Then for all p and m, the group homomorphism*

$$\Sigma^m : \tilde{\mathscr{C}}_p(X, Y) \longrightarrow \tilde{\mathscr{C}}_{p+m}(\Sigma^m X, \Sigma^m Y)$$

induced by complex suspension, is a homotopy equivalence.

If both X and Y are connected, then there are natural isomorphisms

$$\pi_*\big(\tilde{\mathscr{C}}_m(\Sigma^m X, \Sigma^m Y)\big) \cong H_*(X, Y; \mathbf{Z})$$

for all $m \geq 0$.

Each of these statements carries over to algebraic cycles mod k *with the group $H_*(X, Y; \mathbf{Z})$ replaced by $H_*(X, Y; \mathbf{Z}_k)$.*

The long exact sequence of homotopy groups associated to the fibration (1.3) strictly generalizes the long exact sequence in homology associated to the pair (X, Y) (the case where $p = 0$).

In Theorem 3 the requirement that X be an algebraic subvariety is not necessary. The result carries over to any closed subset X for which the topological space $\mathscr{C}_p(X)$ is nice—a countable CW-complex, for example.

Remark. Despite appearances, the arguments given here are essentially algebraic in nature. (See the comments at the end of §4.) At the first writing of this paper the author felt that with the appropriate machinery such as etale homotopy theory [AM], [Fr] and much hard work—Chow varieties are quite difficult to work with—these results should carry over to a quite general algebraic setting. Recently, Eric Friedlander has succeeded in carrying through this program, and has done much more [Fr2], [Fr3]. One of the important features of Friedlander's etale version is that the groups $\pi_*\mathscr{C}_*(X)$ become Galois modules for varieties defined over subfields of the field in question. Friedlander and Barry Mazur [FrM] have also used the complex join construction (cf. §2) to define a functorial algebra of operators on the groups $\pi_*\mathscr{C}_*(X)$ which yield, in particular, a filtration of Hodge type on the integral homology.

Computations of the groups $\pi_*\mathscr{C}_*(X)$ where X is a compact hermitian symmetric space have recently been carried out by P. C. Lima-Filho.

Organization. In Section 2 the fundamental properties of spaces of analytic cycles are presented. In Section 3 an outline of the proof of the main result—the first assertion of Theorem 3—is given. This proof falls into two distinct parts which are presented in Sections 4 and 5. In Section 6 we cover the remaining topics: adapting the arguments given for $\mathscr{C}_p(X)$ to the group $\tilde{\mathscr{C}}_p(X)$; proving Theorem 2; and proving the fibration properties which yield Theorems 5 and 8.

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2. Cycle spaces

We begin by examining the fundamental properties of cycle spaces and their completions. Recall that each $\mathscr{C}_{p,d}(\mathbf{P}^n)$ is canonically an algebraic variety. Therefore, considered as an analytic space, it has a natural topology. This topology can be introduced in many different ways—just like the topology on the space of holomorphic functions where: L^2-convergence on compacta $\Leftrightarrow$ C^0-convergence on compacta $\Leftrightarrow$ C^k-convergence on compacta, etc. Here we shall use definitions of this topology coming from geometric measure theory. We begin by recalling some facts from the theory of currents and analytic varieties. (See [H].)

Let $V \subset \mathbf{P}^n$ be an irreducible subvariety of dimension p with singular set $\operatorname{sing}(V)$. Then $\operatorname{sing}(V)$ is a subvariety of dimension $\leq p-1$, and the set of regular points $\operatorname{reg}(V) = V - \operatorname{sing}(V)$ is a complex p-dimensional submanifold of finite $2p$-measure in $\mathbf{P}^n$. Integration of $2p$-forms over $\operatorname{reg}(V)$ defines an integral current in the sense of Federer, which we denote by (V). This current has no boundary; that is, it satisfies

$$\int_{\operatorname{reg}(V)} d\varphi = 0$$

for all smooth $(2p-1)$-forms φ on $\mathbf{P}^n$. This gives us an embedding

$$(2.1) \qquad \begin{aligned} \mathscr{C}_{p,d}(\mathbf{P}^n) &\longrightarrow \mathscr{Z}_{2p}(\mathbf{P}^n) \\ c = \sum n_\alpha V_\alpha &\hookrightarrow (c) = \sum n_\alpha (V_\alpha) \end{aligned}$$

where $\mathscr{Z}_{2p}(\mathbf{P}^n)$ denotes the set of integral $2p$-cycles on $\mathbf{P}^n$.

There are two topologies on $\mathscr{Z}_{2p}(\mathbf{P}^n)$ which are of interest here. The first is the *weak topology* induced by considering currents as linear functionals on differential forms. A sequence $\{c_n\}_{n=1}^{\infty}$ converges to c weakly in $\mathscr{Z}_{2p}(\mathbf{P}^n)$ if

$$c_n(\varphi) \longrightarrow c(\varphi)$$

for each $2p$-form φ on $\mathbf{P}^n$. The second topology is induced by the *Whitney flat-norm* which is defined by

$$\|c - c'\|_\flat \stackrel{\text{def}}{=} \inf\{M(c - c' - \partial U) + \mathbf{M}(U)\}$$

where the inf is taken over all $(2p+1)$-currents U and where $\mathbf{M}$ denotes the *mass* of the current, defined using the Fubini-Study metric on $\mathbf{P}^n$ (cf. [F]). The following are basic facts in the theory. For $\mu > 0$, set

$$\mathscr{Z}_{k,\mu}(\mathbf{P}^n) = \{c \in \mathscr{Z}_k(\mathbf{P}^n) : \mathbf{M}(c) \leq \mu\}.$$

THEOREM 2.1 (Federer and Fleming [FF]). *For each k and μ, the weak and the flat-norm topologies agree on $\mathscr{Z}_{k,\mu}(\mathbf{P}^n)$, and in this topology the space $\mathscr{Z}_{k,\mu}(\mathbf{P}^n)$ is compact.*

THEOREM 2.2 (J. King [K]). *Let $c \in \mathscr{Z}_{2p}(\mathbf{P}^n)$ be a current which is of type (p, p) and positive. Then $c \in \mathscr{C}_{p,d}(\mathbf{P}^n)$ for some d.*

The degree d in Theorem 2.2 is completely equivalent to the mass since for any positive (p, p)-current $c \in \mathscr{Z}_{2p}(\mathbf{P}^n)$ we have that

$$\mathbf{M}(c) = c\left(\frac{1}{p!}\omega^p\right) = d\int_{\mathbf{P}^p}\frac{1}{p!}\omega^p = d \cdot \mathbf{M}(\mathbf{P}^p) \tag{2.2}$$

where $\mathbf{P}^p$ denotes any linear p-dimensional subspace of $\mathbf{P}^n$ and ω denotes the Kähler form.

Combining Theorems 2.1 and 2.2 gives the following basic result.

THEOREM 2.3 (The Weak Compactness Theorem). *For each p and d the subspace $\mathscr{C}_{p,d}(\mathbf{P}^n) \subset \mathscr{Z}_{2p}(\mathbf{P}^n)$ is compact in the weak ($=$ flat-norm) topology.*

Proof. Consider the sequence $\{c_m\}_{m=1}^{\infty} \subset \mathscr{C}_{p,d}(\mathbf{P}^n)$. Since $\mathbf{M}(c_m) = d\mathbf{M}(\mathbf{P}^p) =$ constant, Theorem 2.1 implies that there exists a subsequence which converges to an element $c \in \mathscr{Z}_{2p}(\mathbf{P}^n)$. The condition of being of type (p, p) and positive is preserved under weak limits. Hence by Theorem 2.2, c is an effective algebraic p-cycle. By definition of weak convergence, we have $c(\omega^p) = \lim_m c_m(\omega^p)$, and so by (2.2) we conclude that degree$(c) = d$. □

Remark 2.4. The flat-norm topology agrees with the standard topology of $\mathscr{C}_{p,d}(\mathbf{P}^n)$ considered as an analytic space. This fact is not used here, so we shall not prove it. However, it follows easily from arguments given in Section 4.

We now fix a distinguished p-dimensional linear subspace $\ell_0 \subset \mathbf{P}^n$ and consider for each $d \geq 1$ the analytic embedding

$$\mathscr{C}_{p,d}(\mathbf{P}^n) \hookrightarrow \mathscr{C}_{p,d+1}(\mathbf{P}^n)$$

defined by $c \mapsto c + \ell_0$. From this sequence of embeddings we can form the union

$$\mathscr{C}_p(\mathbf{P}^n) = \varinjlim_{d} \mathscr{C}_{p,d}(\mathbf{P}^n).$$

Note that $\mathscr{C}_p(\mathbf{P}^n)$ can be obtained from the classical Chow monoid $\mathscr{C}_{p,\cdot}(\mathbf{P}^n) = \coprod_{d\geq 0}\mathscr{C}_{p,d}(\mathbf{P}^n)$ by inverting the element ℓ_0. That is,

$$\mathscr{C}_p(\mathbf{P}^n) = \mathscr{C}_{p,\cdot}(\mathbf{P}^n)/\sim$$

where $c \sim c' \Leftrightarrow c = c' + m\ell_0$ for some $m \in \mathbf{Z}$.

We introduce on $\mathscr{C}_p(\mathbf{P}^n)$ the weak topology for the filtration $\{\mathscr{C}_{p,d}(\mathbf{P}^n)\}_{d=1}^{\infty}$ which is defined as follows. Suppose that C is a set and $\{C_\alpha\}_{\alpha\in A}$ is a family of compact topological spaces, each a subset of C. Assume that: $C = \bigcup_\alpha C_\alpha$, $C_\alpha \cap C_\beta$ is closed in C_α for all α, β, and the topologies induced on $C_\alpha \cap C_\beta$ by C_α and C_β coincide. Then the weak or *weak limit* topology for $\{C_\alpha\}_{\alpha\in A}$ is defined by declaring $F \subset C$ to be closed if and only if $F \cap C_\alpha$ is closed for all α. Note that each inclusion $C_\alpha \subset C$ is a topological embedding. In all cases considered here this topology on C will be Hausdorff. It has the following basic property.

LEMMA 2.5. *For each compact subset $K \subset C$ there exists an α such that $K \subset C_\alpha$.*

For the proof and a general discussion of these topologies see [W, ChI].

Note that addition of cycles is well defined and continuous in $\mathscr{C}_p(\mathbf{P}^n)$. It makes $\{\mathscr{C}_p(\mathbf{P}^n), +\}$ an abelian topological monoid with unit $[\ell_0]$.

As shown in Section 1, we have $\mathscr{C}_0(\mathbf{P}^n) = \mathrm{SP}(\mathbf{P}^n)$ and $\mathscr{C}_{n-1}(\mathbf{P}^n) = \mathbf{P}^\infty$ with weak topologies for the families $\{\mathrm{SP}^d(\mathbf{P}^n)\}_{d=1}^{\infty}$ and $\{\mathbf{P}^d\}_{d=1}^{\infty}$ respectively. However, the spaces $\mathscr{C}_{1,d}(\mathbf{P}^3)$ are complicated, and $\mathscr{C}_1(\mathbf{P}^3)$ is somewhat mysterious. Nevertheless each monoid $\mathscr{C}_p(\mathbf{P}^n)$ has the following elementary properties.

LEMMA 2.6. *For all n and p, the space $\mathscr{C}_p(\mathbf{P}^n)$ is simply-connected. In particular it is connected, and so for any element $c \in \mathscr{C}_p(\mathbf{P}^n)$ the translation τ_c: $\mathscr{C}_p(\mathbf{P}^n) \to \mathscr{C}_p(\mathbf{P}^n)$, given by $\tau_c(c') = c + c'$, is a homotopy equivalence.*

Proof. Let f: $S^1 \to \mathscr{C}_p(\mathbf{P}^n)$ be a continuous map. By Lemma 2.5 there exists an integer d so that $f(S^1) \subset \mathscr{C}_{p,d}(\mathbf{P}^n)$. Since $\mathscr{C}_{p,d}(\mathbf{P}^n)$ is an algebraic variety and therefore triangulable, we may assume by approximation that f is PL. In particular we may assume that $\dim_{\mathbf{R}}(\bigcup_\theta f(\theta)) \le 2p + 1$. It then follows easily from Sard's theorem for families [HL, Appendix A] or from integral geometry [B] that there exists a linear subspace $\ell_0^\wedge \subset \mathbf{P}^n$ of dimension $n - p - 1$ such that

$$\ell_0^\wedge \cap \Big(\ell_0 \cup \bigcup_\theta f(\theta)\Big) = \varnothing .$$

The canonical projection π: $(\mathbf{P}^n - \ell_0^\wedge) \to \ell_0$ makes $(\mathbf{P}^n - \ell_0^\wedge)$ a holomorphic vector bundle over ℓ_0. Let μ_t: $(\mathbf{P}^n - \ell_0^\wedge) \to (\mathbf{P}^n - \ell_0^\wedge)$ denote scalar multiplication by $t \in \mathbf{C}$ in this bundle. Then $f_t = \mu_t \circ f$, for $0 \le t \le 1$, gives a homotopy of f to the constant map $f \equiv d[\ell_0]$. This argument clearly shows that $\mathscr{C}_p(\mathbf{P}^n)$ is connected (take $f =$ constant) and simply-connected. $\square$

Suppose that in defining our monoid we replaced ℓ_0 by a cycle $c \in \mathscr{C}_{p,d}(\mathbf{P}^n)$. Then the resulting space

$$\lim\{ \cdots \hookrightarrow \mathscr{C}_{p,k}(\mathbf{P}^n) \overset{+c}{\hookrightarrow} \mathscr{C}_{p,k+d}(\mathbf{P}^n) \hookrightarrow \cdots \}$$

will be homotopy equivalent to $\mathscr{C}_p(\mathbf{P}^n)$, because c is connected to $d\ell_0$.

We now pass to more general objects. We begin with the disjoint union of spaces

$$\mathscr{C}_{p,\cdot}(\mathbf{P}^n) = \{0\} \amalg \coprod_{d=1}^{\infty} \mathscr{C}_{p,d}(\mathbf{P}^n)$$

which under $+$ becomes an abelian topological monoid with unit $\{0\}$. For $X \subset \mathbf{P}^n$, we say that a cycle c has *support* in X (and write $c \subset X$) if $c = 0$ or $c = \sum n_\alpha V_\alpha$ with $\bigcup_\alpha V_\alpha \subset X$. The set

$$\mathscr{C}_{p,\cdot}(X) = \{c \in \mathscr{C}_{p,\cdot}(\mathbf{P}^n) \colon c \subset X\}$$

is a topological submonoid with the following properties.

PROPOSITION 2.7. *If $X \subset \mathbf{P}^n$ is a closed subset, then $\mathscr{C}_{p,\cdot}(X) \subset \mathscr{C}_{p,\cdot}(\mathbf{P}^n)$ is a closed submonoid. If $X \subset \mathbf{P}^n$ is an algebraic subvariety, then each connected component of $\mathscr{C}_{p,\cdot}(X)$ is an algebraic subvariety of $\mathscr{C}_{p,d}(\mathbf{P}^n)$ for some d.*

Proof. The first statement is a consequence of Proposition 2.1. The second is classical (see [S], [Sh]) and has been established in greater generality by Barlet [B]. □

Remark concerning notation. If $c \in \mathscr{C}_p(\mathbf{P}^n)$ is considered to be a current, the condition "$c \subset A$" would be written "supp$(c) \subset A$". We shall make similar abbreviations here. "$x \in c$" will mean that "$x \in \text{supp}(c)$". If $f\colon U \to X$ is a holomorphic map defined on a neighborhood of supp(x), then $f(c)$ will denote the push-forward $f_\#(c)$ of the current (c), and $c \cap f^{-1}(x)$ will denote the slice $\langle c, f, x\rangle$ of the current at a point x.

Suppose now that $X \subset \mathbf{P}^n$ is an algebraic subvariety and fix $p \geq 0$. Set $A = \pi_0(\mathscr{C}_{p,\cdot}(X))$ and write

$$\mathscr{C}_{p,\cdot}(X) = \coprod_{\alpha \in A} \mathscr{C}_{p,\alpha}(X)$$

as a disjoint union of its connected components. Choose a cross section x of the projection $\mathscr{C}_{p,\cdot}(X) \to A$; that is, choose a cycle $x_\alpha \in \alpha$ for each α. Translation by x_α gives an algebraic embedding $\mathscr{C}_{p,\beta}(X) \xrightarrow{+x_\alpha} \mathscr{C}_{p,\beta+\alpha}(X)$ for each β. If x is a monoid homomorphism, i.e., if $x_{\alpha+\beta} = x_\alpha + x_\beta$ for all α and β, then we can form the direct limit

$$\mathscr{C}_p(X) = \varinjlim_{\alpha} \mathscr{C}_{p,\alpha}(X) \tag{2.3}$$

endowed with the weak topology for the family of subspaces $\{\mathscr{C}_{p,\alpha}(X)\}_{\alpha\in A}$. This directly generalizes the construction for $\mathbf{P}^n$ where $A \cong \mathbf{Z}^+$. Note that $\mathscr{C}_p(X)$ is a connected space with the property that

$$\pi_j(\mathscr{C}_p(X)) \cong \varinjlim_{\alpha} \pi_j(\mathscr{C}_{p,\alpha}(X)). \tag{2.4}$$

For a general cross section x we know only that the translation of $\mathscr{C}_{p,\cdot}(X)$ by $x_{\alpha+\beta}$ is homotopic to translation by $x_\alpha + x_\beta$ for each α and β. Here we take the *Friedlander completion*

$$\mathscr{C}_p(X) = \underset{\alpha}{\underrightarrow{\mathrm{Flim}}}\, \mathscr{C}_{p,\alpha}(X) \tag{2.5}$$

defined as follows. Choose a sequence $\{\alpha_j\}_{j=1}^{\infty}$ in A in which each element of A appears infinitely often. For each j consider the map $\tau_j\colon \mathscr{C}_{p,\cdot}(X) \to \mathscr{C}_{p,\cdot}(X)$ given by $\tau_j(c) = c + x_{\alpha_j}$. We can then take the mapping telescope $\mathrm{Tel}(\mathscr{C}_{p,\cdot}(X), \{\alpha_j\}_{j=1}^{\infty})$ of the sequence of maps:

$$\xrightarrow{\tau_{j-1}} \mathscr{C}_{p,\cdot}(X) \xrightarrow{\tau_j} \mathscr{C}_{p,\cdot}(X) \xrightarrow{\tau_{j+1}} .$$

The Friedlander completion is defined to be the connected component of 0 (at the initial space) in this telescope. This gives a connected **H**-space with property (2.4). In fact is is an infinite loop space. Its homotopy type is independent of the choice of the cross section x and of the sequence τ. A proof of these assertions can be found in [Fr3]. It is property (2.4) that will be crucial for our purposes here.

Now another object natural to consider in this context is the group $\tilde{\mathscr{C}}_p(X)$ of all (not just positive) p-cycles in X. This is the free abelian group generated by the irreducible p-dimensional subvarieties with support in X. There is a natural map

$$\delta\colon \mathscr{C}_{p,\cdot}(X) \times \mathscr{C}_{p,\cdot}(X) \to \tilde{\mathscr{C}}_p(X) \tag{2.6}$$

given by $\delta(c, c') = c - c'$. We introduce on $\tilde{\mathscr{C}}_p(X)$ the weak topology for the family of quotient spaces $F_{\alpha\beta} = \delta(\mathscr{C}_{p,\alpha}(X) \times \mathscr{C}_{p,\beta}(X))$. This is just the quotient topology for δ taken in the compactly generated category. $\tilde{\mathscr{C}}_p(X)$ is an abelian topological group. It is simply the naive topological group completion of $\tilde{\mathscr{C}}_p(X)$ (again in the compactly generated category). A result of Dold and Thom [DT] states that when $p = 0$ and X is connected, the natural embedding $\mathscr{C}_0(X) \hookrightarrow \tilde{\mathscr{C}}_0(X)$ (denoted by $\mathrm{SP}(X) \hookrightarrow AG(X)$ in their paper), is a homotopy equivalence. In general $\mathscr{C}_p(X)$ is homotopy equivalent to the identity component in $\tilde{\mathscr{C}}_p(X)$. This will be discussed elsewhere; here the spaces will be treated separately.

Note that in general $\tilde{\mathscr{C}}_p(X)$ is not connected. In fact $\pi_0\tilde{\mathscr{C}}_p(X)$ is the group completion of the monoid $\pi_0\mathscr{C}_{p,\cdot}(X)$.

Remark 2.8. Given an algebraic subvariety $Y \subset X$, the embedding

$$\mathscr{C}_{p,\cdot}(Y) \subset \mathscr{C}_{p,\cdot}(X)$$

is, at each finite level, an inclusion of algebraic varieties. So also is the embedding

$$k\mathscr{C}_{p,\cdot}(X) \subset \mathscr{C}_{p,\cdot}(X)$$

for each $k > 0$. It follows that the subgroups

$$\tilde{\mathscr{C}}_p(Y) \subset \tilde{\mathscr{C}}_p(X) \quad \text{and} \quad k\tilde{\mathscr{C}}_p(X) \subset \tilde{\mathscr{C}}_p(X)$$

are closed, and the quotients $\tilde{\mathscr{C}}_p(X, Y)$ and $\tilde{\mathscr{C}}_p(X) \otimes \mathbf{Z}_k$ are Hausdorff.

Remark 2.9. Given an algebraic map $f: Y \to X$ between algebraic varieties, there is a natural induced map

$$f_{\#}: \mathscr{C}_{p,\cdot}(Y) \longrightarrow \mathscr{C}_{p,\cdot}(X)$$

which is a continuous monoid homomorphism. One way to construct $f_{\#}$ is to take the induced map on integral currents and then apply the structure theorem 2.2 to conclude that the image lies in $\mathscr{C}_{p,\cdot}(X)$. It follows that there is an induced continuous group homomorphism

$$\tilde{f}_{\#}: \tilde{\mathscr{C}}_p(Y) \longrightarrow \tilde{\mathscr{C}}_p(X).$$

We now introduce some synthetic constructions for spaces of algebraic cycles. Let n and m be non-negative integers and consider a fixed pair of disjoint linear subspaces $\mathbf{P}^n, \mathbf{P}^m \subset \mathbf{P}^{n+m+1}$. Then $\mathbf{P}^{n+m+1}$ can be expressed canonically as the "complex join" of $\mathbf{P}^n$ and $\mathbf{P}^m$, i.e., as the union of all lines joining $\mathbf{P}^n$ to $\mathbf{P}^m$. The linear projections

$$\pi_m: (\mathbf{P}^{n+m+1} - \mathbf{P}^n) \longrightarrow \mathbf{P}^m \quad \text{and} \quad \pi_n: (\mathbf{P}^{n+m+1} - \mathbf{P}^m) \longrightarrow \mathbf{P}^n$$

have the structure of holomorphic vector bundles of rank $(n + 1)$ and $(m + 1)$ respectively.

Definition 2.10. Given closed subsets $A \subset \mathbf{P}^n$ and $B \subset \mathbf{P}^m$, we define the *complex join* of A and B to be the union of all projective lines joining A to B, i.e., the subset

$$A \#_{\mathbf{C}} B = \overline{\pi_n^{-1}(A) \cap \pi_m^{-1}(B)}.$$

In the special case where $B = \mathbf{P}^n$, the set

$$\Sigma^{m+1} A = A \#_{\mathbf{C}} \mathbf{P}^m$$

is called the $(m + 1)$*-fold complex suspension* of A. See Figure 1.

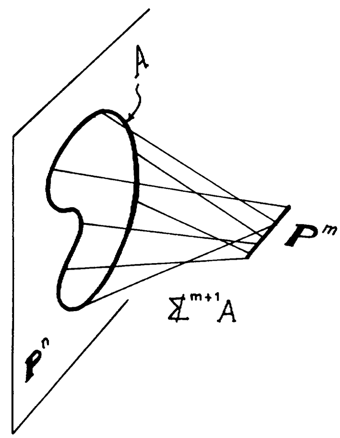

FIGURE 1

The complex join can be viewed in homogeneous coordinates as follows. Given $A \subset \mathbf{P}^n$, set $C(A) = \pi^{-1}(A) \cup \{0\}$ where $\pi: \mathbf{C}^{n+1} - \{0\} \to \mathbf{P}^n$ is the standard projection. Then $C(A \#_{\mathbf{C}} B) = C(A) \times C(B)$.

Note that ΣA is the Thom space of the hyperplane bundle $\mathcal{O}(1)$ restricted to A.

If $X \subset \mathbf{P}^n$ is an algebraic variety, then ΣX is defined by the same polynomial equations that define X, but considered now to be equations in $n + 2$ variables. Fix $\mathbf{P}^n, \mathbf{P}^{m-1} \subset \mathbf{P}^{n+m}$ as above (with m replaced by $m - 1$). Given an algebraic subvariety $X \subset \mathbf{P}^n$ and $p \geq 0$, let

$$\Sigma^m: \mathscr{C}_{p,\cdot}(X) \to \mathscr{C}_{p+m,\cdot}(\Sigma^m X) \tag{2.7}$$

be the continuous monoid homomorphism determined by associating to the irreducible variety V in X the irreducible variety $\Sigma^m V$ in $\Sigma^m X$. At each finite level, Σ^m is an algebraic embedding. It also has the following property which is proved in Section 6.

LEMMA 2.11. *The map* (2.7) *induces a bijection on connected components.*

Consequently if $\{x_\alpha\}_{\alpha \in A}$ is the cross section used to define $\mathscr{C}_p(X)$, then $\{\Sigma^m x_\alpha\}_{\alpha \in A}$ is a cross section that can be used to define $\mathscr{C}_{p+m}(\Sigma^m X)$. Σ^m thereby induces a map

$$\Sigma^m: \mathscr{C}_p(X) \longrightarrow \mathscr{C}_{p+m}(\Sigma^m X). \tag{2.8}$$

Of course by universality the continuous monoid homomorphism (2.7) extends to a continuous group homomorphism

$$(2.9) \qquad \Sigma\!\!\!/^m: \tilde{\mathscr{C}}_p(X) \longrightarrow \tilde{\mathscr{C}}_{p+m}(\Sigma\!\!\!/^m X)$$

which descends to the quotients

$$(2.10) \qquad \Sigma\!\!\!/^m: \tilde{\mathscr{C}}_p(X, Y) \longrightarrow \tilde{\mathscr{C}}_{p+m}(\Sigma\!\!\!/^m X, \Sigma\!\!\!/^m Y),$$

$$(2.11) \qquad \Sigma\!\!\!/^m: \tilde{\mathscr{C}}_p(X) \otimes \mathbf{Z}_k \longrightarrow \tilde{\mathscr{C}}_{p+m}(\Sigma\!\!\!/^m X) \otimes \mathbf{Z}_k$$

for any subvariety $Y \subset X$ and any $k > 0$.

Remark 2.12. Our definition of $\mathscr{C}_p(X)$ above could be enhanced as follows. To each submonoid $\mathscr{U} \subset \pi_0 \mathscr{C}_{p,\cdot}(X)$ we can associate the submonoid

$$\mathscr{C}_{p,\cdot}(X \mid \mathscr{U}) \stackrel{\text{def}}{=} \coprod_{\alpha \in \mathscr{U}} \mathscr{C}_{p,\alpha}(X)$$

and define $\mathscr{C}_p(X \mid \mathscr{U})$ to be the Friedlander completion over $\mathscr{U}$ as above. By Lemma 2.11 there is a natural map $\Sigma\!\!\!/: \mathscr{C}_p(X \mid \mathscr{U}) \to \mathscr{C}_{p+1}(\Sigma\!\!\!/ X \mid \Sigma\!\!\!/ \mathscr{U})$. The arguments given below will carry over without change to prove that this map is a homotopy equivalence.

3. Architectural sketches

We present here the broad outlines of our arguments. Let $X \subset \mathbf{P}^n$ be an algebraic subvariety and fix a linear embedding $\mathbf{P}^n \subset \mathbf{P}^{n+1}$. Choose a point x_0 $(= \mathbf{P}^0) \in \mathbf{P}^{n+1} - \mathbf{P}^n$ and consider the complex suspension maps $\Sigma\!\!\!/: \mathscr{C}_p(X) \to \mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$ defined in Section 2. Our central assertion is the following.

THEOREM 3.1. *For any algebraic subvariety $X \subset \mathbf{P}^n$ the map*

$$\Sigma\!\!\!/: \mathscr{C}_p(X) \longrightarrow \mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$$

is a homotopy equivalence.

It follows immediately that $\Sigma\!\!\!/^m: \mathscr{C}_p(X) \to \mathscr{C}_{p+m}(\Sigma\!\!\!/^m X)$ is a homotopy equivalence for all $m > 0$. Taking $X = \mathbf{P}^q$ and applying the Dold-Thom Theorem prove Theorem 1. Theorem 2 follows from Theorem 1 and known results for $p = 0$ (see §6).

To prove Theorem 3.1 we introduce an intermediate space $\mathscr{T}$ as follows. Embed $X \subset \Sigma\!\!\!/ X$ by taking $X = P^n \cap (\Sigma\!\!\!/ X)$. Say that a cycle $c = \sum_\alpha n_\alpha V_\alpha \in \mathscr{C}_{p+1,\cdot}(\Sigma\!\!\!/ X)$ *has no component contained in* X, and write $c \not\vDash X$, if $V_\alpha \not\subset X$ for each α. Set

$$\mathscr{T} = \left\{ c \in \mathscr{C}_{p+1,\cdot}(\Sigma\!\!\!/ X) : c \not\vDash X \right\}.$$

and let

$$\mathscr{T} = \underset{\alpha}{\underrightarrow{\mathrm{Flim}}}\, \mathscr{T}_\alpha$$

be the Friedlander completion over connected components, defined as above. The proof of Lemma 2.11 shows that the maps $\pi_0\mathscr{C}_{p,\cdot}(X) \to \pi_0\mathscr{T}_{\cdot} \to \pi_0\mathscr{C}_{p+1,\cdot}(\not\Sigma(X))$, induced by suspension and inclusion respectively, are bijections. Therefore in the limit we have canonically defined embeddings

$$\not\Sigma\mathscr{C}_p(X) \subset \mathscr{T} \subset \mathscr{C}_{p+1}(\not\Sigma X).$$

Theorem 3.1 will be proved in the following two steps.

THEOREM 3.2. *The inclusion* $\not\Sigma(\mathscr{C}_p(X)) \subset \mathscr{T}$ *is a homotopy equivalence.*

THEOREM 3.3. *The inclusion* $\mathscr{T} \subset \mathscr{C}_{p+1}(\not\Sigma X)$ *is a homotopy equivalence.*

We shall in fact show that the subset $\not\Sigma(\mathscr{C}_p(X))$ of $\mathscr{T}$ is a deformation retract of $\mathscr{T}$. The second homotopy equivalence is more subtle. It is here that we are forced to access the arbitrarily high degrees available to us in the spaces $\mathscr{C}_{p+1}$.

Theorem 3.1 constitutes the first part of Theorem 3. Its proof is easily adapted to prove also the second part and to prove Lemma 2.9. Details are given in Section 6.

Our second constellation of results is centered on the following result.

THEOREM 3.4. *The surjective homomorphisms given in* (1.2) *and* (1.3) *are principal fibrations.*

In each case complex suspension gives a map of principal fibrations:

$$\begin{array}{ccccc}
\tilde{\mathscr{C}}_p(X) & \xrightarrow{k} & \tilde{\mathscr{C}}_p(X) & \longrightarrow & \tilde{\mathscr{C}}_p(X) \otimes \mathbf{Z}_k \\
\not\Sigma\downarrow & & \not\Sigma\downarrow & & \not\Sigma\downarrow \\
\tilde{\mathscr{C}}_{p+1}(\not\Sigma X) & \xrightarrow{k} & \tilde{\mathscr{C}}_{p+1}(\not\Sigma X) & \longrightarrow & \tilde{\mathscr{C}}_{p+1}(\not\Sigma X) \otimes \mathbf{Z}_k,
\end{array}$$

$$\begin{array}{ccccc}
\tilde{\mathscr{C}}_p(Y) & \xrightarrow{i_*} & \tilde{\mathscr{C}}_p(X) & \longrightarrow & \tilde{\mathscr{C}}_p(X, Y) \\
\not\Sigma\downarrow & & \not\Sigma\downarrow & & \not\Sigma\downarrow \\
\tilde{\mathscr{C}}_{p+1}(\not\Sigma X) & \xrightarrow{i_*} & \tilde{\mathscr{C}}_{p+1}(\not\Sigma Y) & \longrightarrow & \tilde{\mathscr{C}}_{p+1}(\not\Sigma X, \not\Sigma Y)
\end{array}$$

and applying the 5-lemma immediately proves Theorems 5 and 8.

4. Holomorphic taffy

The purpose of this section is to prove Theorem 3.2. Our main tool will be the holomorphic vector field which is zero on $\mathbf{P}^n$ and on its polar point $\mathbf{P}^0$. Its flow φ_t is given as follows. Choose homogeneous coordinates $[z_0, \dots, z_{n+1}]$ for $\mathbf{P}^{n+1}$ so that $\mathbf{P}^n$ corresponds to the hyperplane $z_0 = 0$ and $\mathbf{P}^0 = [1, 0, 0, \dots, 0]$. For $t \in \mathbf{C}^\times$, set

$$\varphi_t([z]) = [tz_0, z_1, \dots, z_{n+1}]. \tag{4.1}$$

We shall work in the following two affine coordinate charts. Let $\xi = (\xi_0, \dots, \xi_n)$ be defined by $\xi_j = z_j/z_{n+1}$ and let $\zeta = (\zeta_1, \dots, \zeta_{n+1})$ be given by $\zeta_j = z_j/z_0$ for each j. Then in these affine coordinates we have that

$$\varphi_t(\xi) = (t\xi_0, \xi_1, \dots, \xi_n) \qquad \text{and} \qquad \varphi_t(\zeta) = \frac{1}{t}\zeta. \tag{4.2}$$

Now for every $t \in \mathbf{C}^\times$ the map $\varphi_t\colon \mathbf{P}^{n+1} \to \mathbf{P}^{n+1}$ is a holomorphic diffeomorphism, and it induces a holomorphic automorphism

$$\varphi_t\colon \mathscr{C}_{p+1,\cdot}(\mathbf{P}^{n+1}) \longrightarrow \mathscr{C}_{p+1,\cdot}(\mathbf{P}^{n+1})$$

which is the identity for $t = 1$. Note that for any $X \subset \mathbf{P}^n$, this flow preserves the subspace $\mathscr{C}_{p+1,\cdot}(\not\Sigma X) \subset \mathscr{C}_{p+1,\cdot}(\mathbf{P}^{n+1})$. Furthermore, it preserves the subspace $\mathscr{T}_\cdot \subset \mathscr{C}_{p+1,\cdot}(\not\Sigma X)$, and for all t it is the *identity* on $\not\Sigma(\mathscr{C}_{p,\cdot}(X)) \subset \mathscr{T}_\cdot$.

Let us restrict φ_t to the real interval $1 \le t < \infty$. Our main assertion is the following.

THEOREM 4.1. *For each $c \in \mathscr{T}_\cdot$, there exists a limit*

$$\varphi_\infty(c) = \lim_{t\to\infty} \varphi_t(c) \in \not\Sigma\big(\mathscr{C}_{p,\cdot}(X)\big) \tag{4.3}$$

which is continuous in c and defines a retraction

$$\varphi_\infty\colon \mathscr{T}_\cdot \longrightarrow \not\Sigma\big(\mathscr{C}_{p,\cdot}(X)\big). \tag{4.4}$$

Furthermore, the extended map

$$\varphi\colon \mathscr{T}_\cdot \times [1, \infty] \longrightarrow \mathscr{T}_\cdot \tag{4.5}$$

is continuous, and therefore (since $\varphi_1 = \mathrm{Id}$), φ_∞ is a deformation retraction.

Proof. For clarity of exposition we shall begin with the case where $p = 0$ and $X = \mathbf{P}^n$. All essential difficulties already reside here. In fact the restriction to general subvarieties $X \subset \mathbf{P}^n$ comes essentially for free.

Fix a 1-cycle $c = \sum n_\alpha V_\alpha \in \mathscr{T}_d$ of degree d and note that $c \cap \mathbf{P}^n$ is a finite set. Hence there exists a linear subspace $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ so that $c \cap \mathbf{P}^{n-1} = \varnothing$. This implies that there exists an $\varepsilon > 0$ so that

$$c \cap V_\varepsilon = \varnothing$$

where $V_\varepsilon = \{x \in \mathbf{P}^{n+1}: \operatorname{dist}(x, \mathbf{P}^{n-1}) \le \varepsilon\}$. After an appropriate linear change of the coordinates $z_0, \ldots, z_{n+1}$ we may assume that $\mathbf{P}^{n-1}$ is given by the equations $z_0 = z_{n+1} = 0$. In the affine coordinates ξ this means that $\mathbf{P}^{n-1}$ is the hyperplane at infinity in the hyperplane $\xi_0 = 0$. Hence after a homothety of coordinates, we may assume the following. Let c_ξ denote that piece of c which lies in the ξ affine coordinate chart. Set

$$\Delta_0 = \{\xi_0 \in \mathbf{C}: |\xi_0| < 1\}, \quad \Delta' = \{\xi' = (\xi_1, \ldots, \xi_n) \in \mathbf{C}^n: \|\xi'\| < 1\},$$

and let $\Delta = \Delta_0 \times \Delta' \subset \mathbf{C}^{n+1}$ be the product neighborhood of the origin. Set $T = \{(\xi_0, \xi') \in \mathbf{C}^{n+1}: \|\xi'\| < |\xi_0|\}$. Then we may assume that

$$c_\xi \subset \Delta \cup T \qquad (4.6)$$

(see Figure 2).

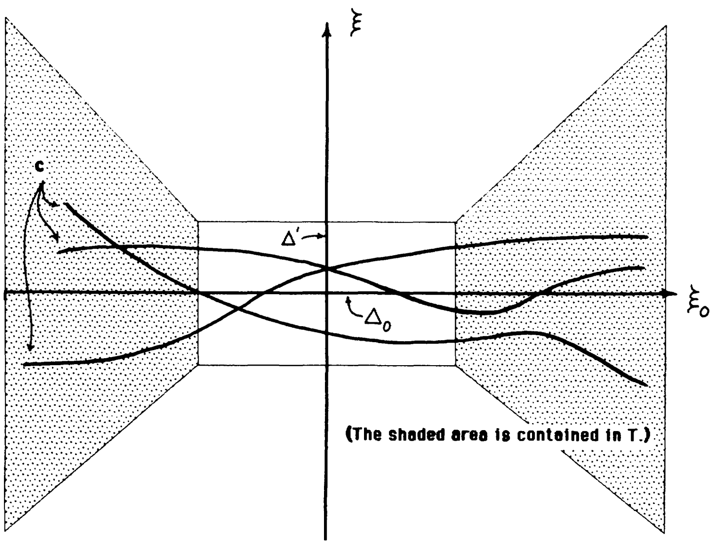

FIGURE 2

We can now decompose c as a current by setting

$$c = c_\Delta + c_{\bar{T}} \qquad (4.7)$$

where c_Δ denotes the restriction of c to the bidisk Δ. Note that

$$\operatorname{supp}(c_{\bar{T}}) \subset \bar{T} \qquad (4.8)$$

where $\bar{T}$ denotes the closure of T in $\mathbf{P}^{n+1}$.

Within the bidisk $\Delta = \Delta_0 \times \Delta'$ the curve c has a canonical presentation as the graph of a holomorphic, d-valued function, i.e., a holomorphic map

$$\sigma_c: \Delta_0 \longrightarrow \mathrm{SP}^d(\Delta') \qquad (4.9)$$

where $SP^d(\Delta')$ denotes the d-fold symmetric product of Δ'. This is a completely standard fact. (See [Wy], [H], or [HL] for example.) The main point is that if $pr\colon \Delta_0 \times \Delta' \to \Delta_0$ denotes projection onto the first factor, then

$$(4.10) \qquad pr\,|_{\mathrm{supp}(c)\cap\Delta}\colon \mathrm{supp}(c) \cap \Delta \longrightarrow \Delta_0 \quad \text{is a proper map.}$$

Therefore the push-forward of the current c_Δ is well-defined, and one sees that

$$(4.11) \qquad pr(c_\Delta) = d\,[\Delta_0].$$

More generally let $f(\xi')$ be any holomorphic function on Δ', considered as a function on $\Delta_0 \times \Delta'$. Let $\psi_i, \dots, \psi_d\colon \Delta_0 \to \Delta'$ be the generically locally defined maps which represent the branches of the curve c_Δ. Then fc_Δ is a current with support in $\mathrm{supp}(c) \cap \Delta$, and we find that

$$(4.12) \qquad pr(fc_\Delta) = \sum_{j=1}^{d} f \circ \psi_j [\Delta_0].$$

where the function $\sum_j f(\psi_j(\xi_0))$ is well-defined and holomorphic in Δ_0.

When $n = 1$, the coordinates of the map $\sigma_c = (\sigma_1, \dots, \sigma_d)$ are the coefficients of the Weierstrass polynomial. They are exactly the elementary symmetric functions in $\psi_1, \dots, \psi_d$. These functions can be computed explicitly from the current c as follows. The Weierstrass polynomial of c_Δ is written as

$$\begin{aligned} p_{c_\Delta}(w, z_0) &= w^d - \sigma_1(\xi_0) w^{d-1} + \cdots + (-1)^d \sigma_d(\xi_0) \\ &= \prod_{j=1}^{d} \big(w - \psi_j(\xi_0)\big). \end{aligned}$$

Fix w with $|w| > 1$ and choose a branch of $\log(w - \xi_1)$ in the disk $|\xi_1| < 1$. Then from (4.12) we have

$$(4.13) \qquad pr(\log(w - \xi_1) c_\Delta) = \log p_{c_\Delta}(w, \cdot)[\Delta_0].$$

Taking d/dw we obtain the standard Newton identities which relate $\sigma_1, \dots, \sigma_d$ to the "trace power" functions $\tau_r(\xi_0) = \sum_{j=1}^d \psi_j(\xi_0)^r$, which are also explicitly computed from c_Δ by the formula

$$(4.14) \qquad pr(\xi_1^r c_\Delta) = \tau_r [\Delta_0].$$

The main observation here is the following. Let us denote by $\mathrm{Hol}(\Delta_0, SP^d(\Delta'))$ the space of holomorphic maps from Δ_0 to the analytic space $SP^d(\Delta')$, equipped with the strong topology of uniform convergence (and uniform convergence of any finite number of derivatives).

PROPOSITION 4.2. *There is a neighborhood U of c in $\mathscr{C}_{1,d}(\mathbf{P}^{n+1})$ in which the map*

$$\sigma\colon U \to \mathrm{Hol}\big(\Delta_0, SP^d(\Delta')\big)$$

given by $c \to \sigma_c$, is defined and continuous.

Proof. We may take U to be the open subset of all $c' \in \mathscr{C}_{1,d}(\mathbf{P}^{n+1})$ such that $\operatorname{supp}(c') \subset \operatorname{interior}(\Delta \cup \bar{T})$. The map σ is well defined on this set since the condition (4.10) holds for all such curves c'. In the case $n = 1$, the continuous dependence of $\sigma_{c'}$ on c' is a consequence of formulas (4.13) and (4.14). To see this, choose $c', c'' \in U$ and let τ_r', τ_r'' be the functions associated via (4.14). Then it is straightforward to check the following. Given an open set $G \subset \mathbf{P}^{n+1}$ and an integral 2-current e, we define

$$\|e\|_{G,\flat} = \inf\{\mathbf{M}_G(e - \partial e') + \mathbf{M}_G(e')\}$$

where the inf is taken over 3-currents e' and where for any current T, $\mathbf{M}_G(T)$ denotes the mass of T on G; i.e., $\mathbf{M}_G(T) = \mathbf{M}_G(\chi_G T)$ where χ_G is the characteristic function of G. Now it is straightforward to check that

$$\begin{aligned}
\int_{\Delta_0} |\tau_r' - \tau_r''| &= \|(\tau_r' - \tau_r'')\|_{\Delta_0,\flat} \\
&= \|pr_*(\xi_1^r c_\Delta' - \xi_1^r c_\Delta'')\|_{\Delta,\flat} \\
&\le \Big(\sup_\Delta |\xi_1^r|\Big)\Big(\sup_\Delta |\operatorname{Lip}(pr)|^2 + \sup_\Delta |\operatorname{Lip}(pr)|^3\Big)\|c_\Delta' - c_\Delta''\|_{\Delta,\flat} \\
&\le 2\|c_\Delta' - c_\Delta''\|_{\Delta,\flat} \\
&\le 2\|c_\Delta' - c_\Delta''\|_\flat
\end{aligned}$$

(where Lip denotes the standard Lipschitz norm). For each compact set $K \subset \Delta_0$ and each integer $k \ge 0$, there is a constant $\gamma_{K,k}$ so that

$$\sup_K \sum_{\alpha=0}^{k} |D^\alpha \tau| \le \gamma_{K,k} \int_{\Delta_0} |\tau|$$

for all holomorphic functions τ on Δ_0 (by the Cauchy integral formula). Combining the last two statements and using the Newton identities prove the desired uniform convergence on any fixed compact subset of Δ_0. Without loss of generality we may restrict to a disk Δ_0 of smaller radius, and so the proposition is proved when $n = 1$.

When $n > 1$, we reduce to the case where $n = 1$ by considering all projections of Δ' onto 1-dimensional subdisks through the origin. This amounts to replacing ξ_1 with an arbitrary linear function $\lambda = a_1\xi_1 + \cdots + a_n\xi_n$ and proceeding as before to consider the currents

$$pr(\lambda^r c) = \tau_{r,\lambda}[\Delta_0]$$

for $r \in \mathbf{Z}^+$. For each projection we produce a set of Weierstrass coefficients $\sigma_{1,\lambda}, \ldots, \sigma_{d,\lambda}$ which are holomorphic in ξ_0 and vary continuously over U as above. In a neighborhood of any point of $SP^d(\Delta')$, a finite number of coordinates $\sigma_{j,\lambda}$ generate the ring of holomorphic functions, and so the proposition is proved. □

Proposition 4.2 has the following immediate consequence.

COROLLARY 4.3. *The mapping*

$$\rho: \mathscr{T}_d \longrightarrow \mathscr{C}_{0,d}(\mathbf{P}^n)$$

which assigns to each $c \in \mathscr{T}_d$ *the intersection* $\rho(c) = c \cap \mathbf{P}^n$ *(with points counted to multiplicity) is continuous.*

Proof. Fix $c \in \mathscr{T}_d$ and choose U as in Proposition 4.2. Then in U the mapping ρ is given by

$$c \longrightarrow \sigma_c(0) = \rho(c).$$

The continuity of ρ therefore follows from the continuity of σ with the uniform norm on $\mathrm{Hol}(\Delta_0, SP^d(\Delta'))$. □

COROLLARY 4.4. *The mapping*

$$\varphi_\infty: \mathscr{T}_d \longrightarrow \not{\Sigma}(\mathscr{C}_0(\mathbf{P}^n))$$

defined by $\varphi_\infty = \not{\Sigma} \circ \rho$ *is a continuous retraction.*

Our final task is to prove the continuity of the map (4.5). However, before proceeding to the general argument we shall present a completely elementary proof of the weaker statement (4.3). This elementary argument is unnecessary for establishing the main theorem, but it does impart insights into what is taking place in the limiting process.

We want to show that $\lim_{t\to\infty} \varphi_t(c) = \varphi_\infty(c)$. For this we consider the decomposition $c = c_\Delta + c_{\overline{T}}$ given in (4.7) and prove first of all that $\lim_{t\to\infty} \varphi_t(c_{\overline{T}}) = 0$. In fact we shall show directly that

$$\lim_{t\to\infty} \mathbf{M}(\varphi_t(c_{\overline{T}})) = 0 \tag{4.15}$$

which, since the mass-norm dominates the flat-norm, will certainly suffice.

To demonstrate (4.15) we work in the affine coordinates $\zeta = (\zeta_1, \dots, \zeta_{n+1})$ and recall that the current $c_{\overline{T}}$ is supported in the compact subset $K = \overline{T} - \Delta$ of the ζ-coordinate plane. (In fact, $\mathrm{supp}(c'_{\overline{T}}) \subset K$ for all $c' \in U$.) Now the following lemma is an immediate consequence of the area formula. (See [F, 3.2].)

LEMMA 4.5. *For each* $r > 0$ *there is a constant* α_r *so that*

$$\mathbf{M}(\varphi_t Y) \le \frac{\alpha_r}{t^k}\mathbf{M}(Y)$$

for all integral k-currents Y with support in $\{\zeta \in \mathbf{C}^{n+1}: \|\zeta\| \le r\}$ *and for all* $t \ge 1$.

Note that with the euclidean metric on $\mathbf{C}^{n+1}$ we could take $\alpha_r = 1$ for any r. However, for the Fubini-Study metric this is not possible and it is important that $c_{\bar{T}}$ have compact support in $\mathbf{C}^{n+1}$. See Figure 3.

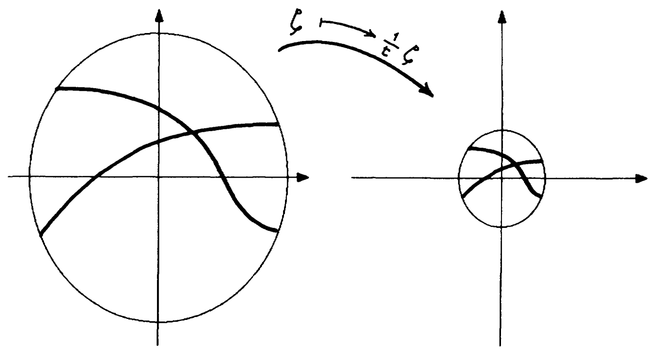

FIGURE 3

Lemma 4.5 immediately implies (4.15). Consequently to establish the first equation (4.3) it will suffice to prove

$$\lim_{t\to\infty} \varphi_t(c_\Delta) = \varphi_\infty(c). \tag{4.16}$$

For this purpose we work in the affine ξ-coordinates where, setting $\xi' = (\xi_1, \ldots, \xi_n)$ as before, we have

$$\varphi_t(\xi_0, \xi') = (t\xi_0, \xi') \qquad \text{(see Figure 4)}.$$

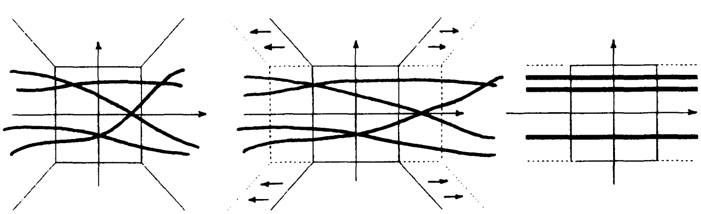

FIGURE 4

We have seen that the current c_Δ is the graph of the holomorphic multivalued function σ_c over the disk Δ_0. It is evident that the current $\varphi_t(c_\Delta)$ is then just the graph of the function

$$\sigma_c^t(\xi_0) \equiv \sigma_c\left(\frac{1}{t}\xi_0\right)$$

defined over the expanded disk

$$\Delta_0(t) = \{\xi_0 \in \mathbf{C}: |\xi_0| < t\}.$$

Using the Fubini-Study metric

$$ds^2 = \frac{1}{\left(1 + \|\xi\|^2\right)^2}\left\{(1 + \|\xi\|^2)\|d\xi\|^2 - |(\xi, d\xi)|^2\right\}$$

one can show explicitly that $\|\varphi_t(c_\Delta) - \varphi_\infty(c)\|_b \to 0$.

Carrying through this argument leads to a good understanding of the nature of this limiting process. However, we shall omit the remaining details because there exists a more elegant, indirect argument which uses the compactness theorem. This indirect argument can be used in fact to prove the joint continuity of the map φ in the variables c and t.

We proceed as follows. Fix a point $(c, \infty) \in \mathcal{T}_d \times [0, \infty]$, and note that to prove the continuity of φ at (c, ∞) it suffices to prove the following assertion:

(4.17) For every sequence $(c_j, t_j) \to (c, \infty)$ there exists a subsequence $(c_{j'}, t_{j'})$ such that

$$\lim_{j' \to \infty} \varphi_{t_{j'}}(c_{j'}) = \varphi_\infty(c).$$

Now given any sequence $(c_j, t_j) \to (c, \infty)$ we know from the Weak Compactness Theorem 2.3 that there exists a subsequence, also denoted (c_j, t_j), and a cycle $\tilde{c} \in \mathscr{C}_{1,d}(\mathbf{P}^{n+1})$ such that

$$\lim_{j \to \infty} \varphi_{t_j}(c_j) = \tilde{c}.$$

It remains to prove that $\tilde{c} = \varphi_\infty(c)$. Let U be the neighborhood of c given in Proposition 4.2. We may assume without loss of generality that $c_j \in U$ for all j, and that the sequence $\{t_j\}$ is monotone increasing. We have shown that inside the set $\Delta_0(t_j) \times \mathbf{C}^n \subset \mathbf{C}^{n+1}$, the cycle $\varphi_{t_j}(c_j)$ is the graph of the multivalued holomorphic function

$$\sigma_{c_j}(\xi_0/t_j).$$

By Proposition 4.2 we know that the functions σ_{c_j} are converging uniformly to the function σ_c over the disk Δ_0. Consequently we have that

$$(4.18) \quad \sup_{|\xi_0| < t_j} \operatorname{dist}\left(\sigma_{c_j}(\xi_0/t_j), \sigma_c(\xi_0/t_j)\right) = \sup_{|\xi_0| < 1} \operatorname{dist}\left(\sigma_{c_j}(\xi_0), \sigma_c(\xi_0)\right) \xrightarrow[j \to \infty]{} 0$$

where dist denotes any metric defining the topology on $\mathrm{SP}^d(\Delta')$. Let us now fix a radius $\rho > 0$ and note that by continuity

$$(4.19) \quad \sup_{|\xi_0| < \rho} \operatorname{dist}\left(\sigma_c(\xi_0/t_j), \sigma_c(0)\right) \xrightarrow[j \to \infty]{} 0.$$

It follows from (4.18) and (4.19) that for any $\rho > 0$,

$$(4.20) \quad \sup_{|\xi_0| < \rho} \operatorname{dist}\left(\sigma_{c_j}(\xi_0/t_j), \sigma_c(0)\right) \xrightarrow[j \to \infty]{} 0.$$

For $\rho > 0$, set $A_\rho = \Delta_0(\rho) \times \mathbf{C}^n \subset \mathbf{C}^{n+1}$, and for any $c \in \mathscr{C}_{1,d}(\mathbf{P}^{n+1})$ let $c \cap A_\rho$ denote the restriction of c to A_ρ. Then (4.20) means exactly that

$$\lim_{j\to\infty} \varphi_{t_j}(c_j) \cap A_\rho = \varphi_\infty(c) \cap A_\rho \tag{4.21}$$

for all $\rho > 0$. It follows that in the affine coordinate chart $\mathbf{C}^{n+1}$ we have

$$\tilde{c} \cap \mathbf{C}^{n+1} = \varphi_\infty(c) \cap \mathbf{C}^{n+1} \cong \varphi_\infty(c)$$

and so

$$\tilde{c} = \varphi_\infty(c) + \tilde{c}' \tag{4.22}$$

where $\operatorname{supp}(\tilde{c}') \subset \mathbf{P}^{n+1} - \mathbf{C}^{n+1}$. However, for positive $(1,1)$-currents γ the mass is given by integrating the Kähler form ω, i.e., $\mathbf{M}(\gamma) = \gamma(\omega)$. Consequently on positive $(1,1)$-currents the mass is *additive* and also *weakly continuous*. Since the Kähler form is closed, we have that $\mathbf{M}(c') = d\mathbf{M}(\mathbf{P}^1) = 4\pi d$ for all curves $c' \in \mathscr{C}_{1,d}(\mathbf{P}^{n+1})$. It follows therefore from (4.22) that

$$4\pi d = \mathbf{M}(\tilde{c}) = \mathbf{M}(\varphi_\infty(c)) + \mathbf{M}(\tilde{c}') = 4\pi d + \mathbf{M}(\tilde{c}')$$

and so $\tilde{c}' = 0$. This establishes assertion (4.17), and the proof of Theorem 4.1 is complete in the case $p = 0$ and $X = \mathbf{P}^n$.

The case for a general subvariety $X \subset \mathbf{P}^n$ now follows immediately by simply restricting the map φ.

The argument for the case where $p > 1$ follows exactly the same arguments as those given above. The only special fact that we have used concerning curves is that they meet $\mathbf{P}^n$ in a finite set of points. For $c \in \mathscr{T}_d \subset \mathscr{C}_{p+1,d}(\mathbf{P}^{n+1})$ we have $c \cap \mathbf{P}^n \in \mathscr{C}_{p,d}(\mathbf{P}^n)$. Given any such c, there exists a linear subspace $\mathbf{P}^{n-p-1} \subset \mathbf{P}^n$ such that $c \cap \mathbf{P}^{n-p-1} = \varnothing$. Hence there is a neighborhood V of $\mathbf{P}^{n-p-1}$ in $\mathbf{P}^{n+1}$ such that $c \cap V = \varnothing$. Choose a linear subspace $\mathbf{P}^p \subset \mathbf{P}^n - V$ and note that without loss of generality we may assume $\mathbf{P}^n - \overline{V}$ to be a normal tube of any desired width $\eta > 0$ about $\mathbf{P}^p$; i.e., we may assume $\mathbf{P}^n - \overline{V} = N_\eta \stackrel{\text{def}}{=} \{x \in \mathbf{P}^n \colon \operatorname{dist}(x, \mathbf{P}^p) < \eta\}$. We may further assume our neighborhood V to have the following form. Given any set $A \subset \mathbf{P}^n$ and any $\varepsilon > 0$, let us define $\Sigma\!\!\!/_\varepsilon(A) = \{x \in \Sigma\!\!\!/(A) \subset \mathbf{P}^{n+1} \colon \operatorname{dist}(x, \mathbf{P}^n) < \varepsilon\}$. Then we may assume that

$$V = \Sigma\!\!\!/_\varepsilon(\mathbf{P}^n - N_\eta)$$

for some $\varepsilon > 0$. Consider now the neighborhood

$$\Delta = \Sigma\!\!\!/_\varepsilon(N_\eta)$$

of $\mathbf{P}^p$ in $\mathbf{P}^{n+1}$, and set

$$\Delta_0 = \Sigma\!\!\!/_\varepsilon(\mathbf{P}^p).$$

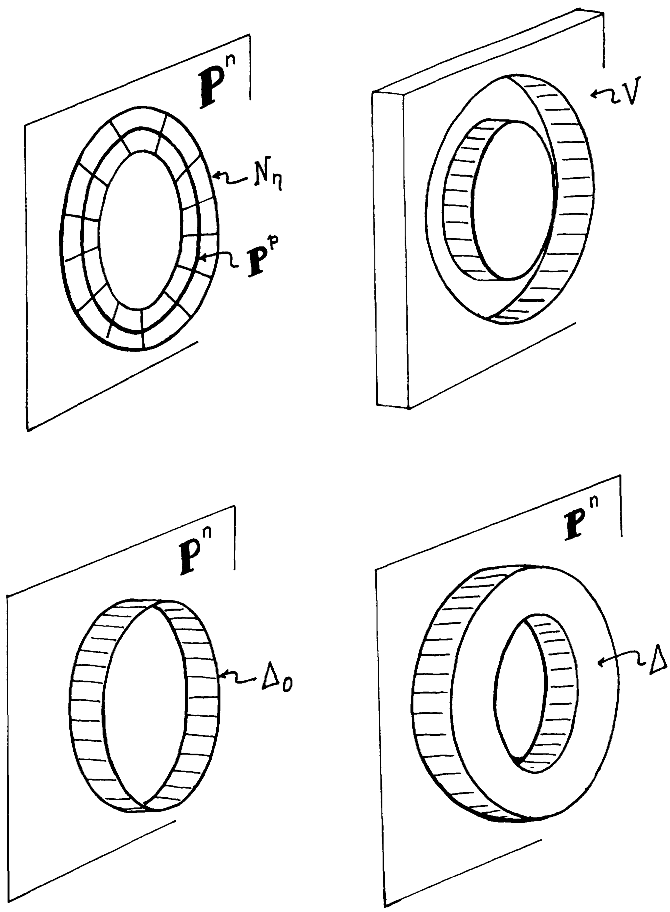

FIGURE 5

There is a natural projection

$$pr\colon \Delta \to \Delta_0$$

and it has the property (4.10) that the *restriction of pr to* $\operatorname{supp}(c) \cap \Delta$ *is a proper map*.

The arguments given above now generalize immediately to this case. To do the analysis one can restrict to some fixed finite cover $\{U_\alpha\}_{\alpha=1}^{M}$ of $\mathbf{P}^p$ by local coordinate charts. Note that $\Delta_0 \to \mathbf{P}^p$ is a holomorphic disk bundle, and so we get a corresponding covering $\{U_{0,\alpha}\}_{\alpha=1}^{M}$ of Δ_0 by product charts $U_{0,\alpha} = U_\alpha \times \{z \in \mathbf{C} : |z| < 1\}$. Similarly, we get a covering $\{U_{0,\alpha} \times \Delta'\}_{\alpha=1}^{M}$ of Δ by product charts, where $\Delta' = \{z \in \mathbf{C}^{n-p}\colon \|z\| < 1\}$. All arguments now go through directly. This proves Theorem 4.1. $\square$

Note. It is important in the construction that we are pulling away from a *hyperplane*. Pulling away from a linear subspace of higher codimension does not extend continuously in the limit to very large subspaces of cycles. Note for example that in a neighborhood of $\mathbf{P}^0$, $\lim_{t \to 0} \varphi_t(c) =$ the tangent cone on c at $\mathbf{P}^0$, which is a very erratically behaved function of c.

Proof of Theorem 3.2. One checks that the map (4.5) canonically determines a map

$$\varphi: \mathscr{T} \times [1, \infty] \to \mathscr{T} \tag{4.23}$$

which is the identity on $\not{\Sigma}(\mathscr{C}_p(X)) \times [1, \infty]$. This follows from the functoriality of the Friedlander completion or simply from direct inspection. □

Remark 4.6. The arguments in this section appear quite analytic in nature. However, closer examination shows that these arguments can very likely be made *purely algebraic*. Let $\mathbf{C} = \mathbf{C}^\times \cup \{\infty\}$ and consider the algebraic map

$$\rho: \mathscr{C}_{p+1,d} \times \mathbf{C}^\times \to \mathscr{C}_{p+1,d}$$

defined as above. Think of φ as "densely defined" in $\mathscr{C}_{p+1,d} \times \mathbf{C}$. In particular, let $\overline{\Gamma_\varphi}$ denote the Zariski closure of the graph of φ in the product $(\mathscr{C}_{p+1,d} \times \mathbf{C}) \times \mathscr{C}_{p+1,d}$. The main point is to show that $\overline{\Gamma_\varphi}$ is single-valued over the Zariski open subset $\mathscr{T}_d \times \mathbf{C} \subset \mathscr{C}_{p+1,d} \times \mathbf{C}$.

It has been pointed out by the referee that indeed this construction is an example of "pulling to the normal cone" (cf. [Fu]).

The arguments given above carry over to cyclic spaces on any sufficiently positive line bundle over a compact Kähler manifold.

5. Magic fans

The purpose of this section is to prove Theorem 3.3. To do this it will suffice to show that the homomorphism

$$\pi_m(\mathscr{T}) \xrightarrow{i_*} \pi_m\big(\mathscr{C}_{p+1}(\not{\Sigma}X)\big) \tag{5.1}$$

induced by the inclusion i: $\mathscr{T} \subset \mathscr{C}_{p+1}(\not{\Sigma}X)$, is an isomorphism for all $m > 0$. To prove this it suffices to establish the following.

(5.2) For any $\beta \in \pi_m(\mathscr{C}_{p+1}(\not{\Sigma}X))$, there exists an integer d_β such that for each integer $d \geq d_\beta$, there is an element $\alpha_d \in \pi_m(\mathscr{T})$ with $i_*(\alpha_d) = d \cdot \beta$.

(5.3) For any $\alpha \in \pi_m(\mathscr{T})$, such that $i_*\alpha = 0$, there exists an integer d_α such that $d \cdot \alpha = 0$ for all $d \geq d_\alpha$.

These assertions are immediate consequences of the following geometric statements.

(5.2′) For any map $f\colon S^m \to \mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$, there exists an integer d_f such that for each $d \geq d_f$ the map $d\cdot f\colon S^m \to \mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$ is homotopic to a map $\tilde{f}\colon S^m \to \mathscr{T} \subset \mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$.

(5.3′) For any map of pairs $f\colon (D^{m+1}, S^m) \to (\mathscr{C}_{p+1}(\Sigma\!\!\!/ X), \mathscr{T})$ there exists an integer d_f such that for each $d \geq d_f$ the map $d\cdot f$ is homotopic through maps of pairs to a map $\tilde{f}\colon (D^{m+1}, S^m) \to (\mathscr{T}, \mathscr{T})$.

Note 5.1. By $d\cdot f$ we mean the map $d\cdot f(x) = f(x) + \cdots + f(x)$ (d-times) where $+$ is the addition in the monoid $\mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$. By replacing f by $d\cdot f$ we are simply raising the multiplicity of each cycle $f(x)$ by a factor of d. One sees easily that

$$[df] = d[f] \quad \text{in} \quad \pi_m\big(\mathscr{C}_{p+1}(\Sigma\!\!\!/ X)\big).$$

Note 5.2. Given any map $f\colon S^m \to \mathscr{C}_p(\mathbf{P}^n)$ there exists an integer d so that $f(S^m) \subset \mathscr{C}_{p,d}(\mathbf{P}^n)$ by Lemma 2.5. Furthermore the spaces

$$\mathscr{C}_{p,d}(X) \overset{\text{def}}{=} \mathscr{C}_{p,\cdot}(X) \cap \mathscr{C}_{p,d}(\mathbf{P}^n) \subset \mathscr{C}_{p,d}(\mathbf{P}^n),$$

where $X \subset \mathbf{P}^n$ is an algebraic subvariety, are all triangulable. Hence, in statements (5.2′) and (5.3′) above we may assume that the given maps f are, say, PL-maps with respect to some triangulation compatible with the canonical smooth stratification of $\mathscr{C}_{p,d}(X)$.

Note 5.3. Before entering into details it may be enlightening to examine the geometric motivation behind the arguments. Essentially we want to show that any finite subcomplex of $\mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$ can be deformed into $\mathscr{T}$. In smooth manifolds this could be accomplished by transversality arguments. Of course our space $\mathscr{C} = \mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$ is far from smooth, and the "bad set" $\mathscr{B} = \mathscr{C} - T$ is always in a highly singular locus. Nevertheless, if we can find subcomplexes of the form $\mathscr{B} \times D^N \subset \mathscr{C}$, with $\mathscr{B} = \mathscr{B} \times \{0\}$ and N arbitrarily large, then we can easily move any complex away from $\mathscr{B}$. As the reader will see, such product complexes do exist in a natural form.

We now present our basic method of constructing the desired homotopies. Consider a fixed hyperplane $\mathbf{P}^n \subset \mathbf{P}^{n+1}$ and a distinguished point $\mathbf{P}^0 = x_\infty \in \mathbf{P}^{n+1} - \mathbf{P}^n$ as above. Projecting away from x_∞ onto $\mathbf{P}^n$ gives a canonical identification

$$\mathbf{P}^{n+1} - \{x_\infty\} \cong \nu$$

where

(5.4) $$\pi: \nu \longrightarrow \mathbf{P}^n$$

is the normal bundle to $\mathbf{P}^n$ in $\mathbf{P}^{n+1}$. The bundle ν is holomorphically equivalent to the hyperplane bundle; i.e., $\nu \cong \mathcal{O}(1)$.

Let $\mathrm{Div}_d = \mathscr{C}_{n,d}(\mathbf{P}^{n+1})$ be the set of effective divisors of degree d on $\mathbf{P}^{n+1}$, and for each d consider the subset

(5.5) $$\mathrm{Div}'_d \stackrel{\mathrm{def}}{=} \{D \in \mathrm{Div}_d: D \subset \nu\}$$

of those divisors which do not contain the point x_∞. Note that Div'_d is exactly the space of holomorphic d-valued sections of the bundle $\nu = \mathcal{O}(1)$. In particular, for any $D \in \mathrm{Div}'_d$, we have $\pi(D) = d \cdot \mathbf{P}^n$, and for any $D \in \mathrm{Div}'_d$ and any point $x \in \mathbf{P}^n$ we have a well-defined divisor

(5.6) $$F_D(x) = D \cap \pi^{-1}(x)$$

of degree d in the fibre $\pi^{-1}(x)$. See Figure 6.

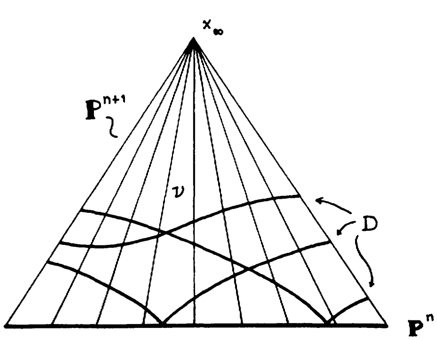

FIGURE 6

With respect to any local holomorphic trivialization $\pi^{-1}(U) \cong U \times \mathbf{C}$ of ν over an open set $U \subset \mathbf{P}^n$, we see that F_D gives a well-defined holomorphic map

$$F_D: U \to \mathscr{C}_{0,d}(\mathbf{C}).$$

The space Div'_d is actually a vector space of dimension $\binom{n+1+d}{d}$. There is a natural holmorphic map $\mathrm{Div}'_d \times \mathbf{C} \to \mathrm{Div}'_d$ given by

(5.7) $$(D, t) \longrightarrow tD \stackrel{\mathrm{def}}{=} \mu_t(D)$$

where $\mu_t: \nu \to \nu$ is scalar multiplication by t in the bundle ν. Note that if

$F_D(x) = \sum n_\alpha p_\alpha$ for $p_\alpha \in \pi^{-1}(x)$, then

$$F_{tD}(x) = \begin{cases} \sum n_\alpha t p_\alpha, & \text{for all } t \neq 0 \\ d \cdot \{0\}, & \text{if } t = 0. \end{cases} \tag{5.8}$$

Suppose now that $V \subset \mathbf{P}^n$ is an irreducible algebraic subvariety of dimension p. The restriction of F_D to V gives us a d-valued section of ν over V whose graph is the holomorphic p-chain

$$\varphi_D(V) \stackrel{\text{def}}{=} D \cap \pi^{-1}(V).$$

This current can be defined analytically by taking the graph of F_D over reg(V). This current has finite mass and no boundary (since its boundary is flat and supported in a subset of Hausdorff dimension $\leq p$). We then apply the Structure Theorem 2.3 to conclude that it is an analytic subvariety. There are many other ways to see that this is well-defined (cf. [Fu]). Note that

$$\deg \varphi_D(V) = d \cdot \deg V$$

where d is the degree of the divisor D. This definition extends immediately to any effective cycle $c = \sum n_\alpha V_\alpha$ in $\mathbf{P}^n$ by setting

$$\varphi_D(c) \stackrel{\text{def}}{=} D \cap \pi^{-1}(c) = \sum n_\alpha \left(D \cap \pi^{-1}(V_\alpha)\right),$$

and again

$$\deg \varphi_D(c) = d \cdot \deg c.$$

Note the special case where $D_0 = d \cdot \mathbf{P}^n$. Here $F_{D_0}(x) = d \cdot \{0\}$ and $\varphi_{D_0}(c) = d \cdot c$ for all $c \in \mathscr{C}_{r,\cdot}(\mathbf{P}^n)$. We are led to consider the subsets $d \cdot \mathscr{C}_{r,d_0}(\mathbf{P}^n) = \{d \cdot c : c \in \mathscr{C}_{r,d_0}(\mathbf{P}^n)\}$.

LEMMA 5.4 (The lifting lemma). *For all r, d_0 and d, the map*

$$d \cdot \mathscr{C}_{r,d_0}(\mathbf{P}^n) \times \mathrm{Div}_d' \xrightarrow{\varphi} \mathscr{C}_{r,dd_0}(\mathbf{P}^{n+1})$$

defined by

$$(d \cdot c, D) \longrightarrow \varphi_D(c)$$

is continuous. For any given $D \in \mathrm{Div}_d'$, the family

$$\varphi_{tD} \colon d \cdot \mathscr{C}_{r,d_0}(\mathbf{P}^n) \longrightarrow \mathscr{C}_{r,dd_0}(\mathbf{P}^{n+1})$$

for $0 \leq t \leq 1$ (where $0 \cdot D = D_0 = d \cdot \mathbf{P}^n$) is a homotopy of φ_D to the inclusion map $d \cdot \mathscr{C}_{r,d_0}(\mathbf{P}^n) \subset \mathscr{C}_{r,dd_0}(\mathbf{P}^{n+1})$.

Proof. To prove that φ is continuous at a point $(d \cdot c, D)$, consider a convergent sequence $(c_j, D_j) \to (c, D)$. By the compactness Theorem 2.3 every subsequence has a sub-subsequence $\{(c_{j'}, D_{j'})\}$ such that

$$\varphi_{D_{j'}}(c_{j'}) \longrightarrow \tilde{c}$$

for some $\tilde{c} \in \mathscr{C}_{r,\,dd_0}(\mathbf{P}^{n+1})$. From the structure of the map φ and the continuity of mass it follows that $\tilde{c} = \varphi_D(c)$. To see this note first that $\operatorname{supp}(\tilde{c}) \subset \operatorname{supp}(\varphi_D(c))$, and so $\tilde{c} = \sum n'_\alpha V_\alpha$ where $\varphi_D(c) = \sum n_\alpha V_\alpha$. Then since global convergence implies local convergence at regular points, we conclude that $n_\alpha = n'_\alpha$ for each α. This proves the continuity of φ. The remainder of the lemma is obvious. □

We apply the lifting construction as follows. Fix a linear subspace $\mathbf{P}^n \subset \mathbf{P}^{n+1}$ and choose a point $x_0 = \mathbf{P}^0 \in \mathbf{P}^{n+1} - \mathbf{P}^n$. This gives

$$\nu_0 \overset{\text{def}}{=} \mathbf{P}^{n+1} - \{x_0\}$$

the structure of a holomorphic line bundle $\pi_0\colon \nu_0 \to \mathbf{P}^n$ as above.

We now embed $\mathbf{P}^{n+1}$ linearly into $\mathbf{P}^{n+2}$ and choose a point

$$x_\infty \in \mathbf{P}^{n+2} - \mathbf{P}^{n+1}.$$

Set $\nu_\infty \overset{\text{def}}{=} \mathbf{P}^{n+2} - \{x_\infty\}$ and let $\pi_\infty\colon \nu_\infty \to \mathbf{P}^{n+1}$ be the corresponding line bundle. It is in this space that we shall do our lifting. Given any divisor $D \in \operatorname{Div}'_d$ on $\mathbf{P}^{n+2}$ and any cycle $c \in \mathscr{C}_{p+1,\,d_0}(\mathbf{P}^{n+1})$ we get a lifting $\varphi_D(c)$ of the cycle $d \cdot c$ into $\mathbf{P}^{n+2}$. See Figure 7.

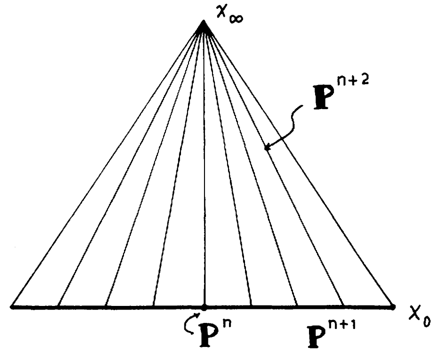

FIGURE 7

Consider now a point x_1 on the line $\overline{x_0 x_\infty}$ which is distinct from x_0 and x_∞, and let

$$\pi_1\colon \left(\mathbf{P}^{n+2} - \{x_1\}\right) \longrightarrow \mathbf{P}^{n+1}$$

be the linear projection away from x_1 onto $\mathbf{P}^{n+1}$. (Think of x_1 as being close to x_∞ even though projectively this is meaningless.) See Figure 8.

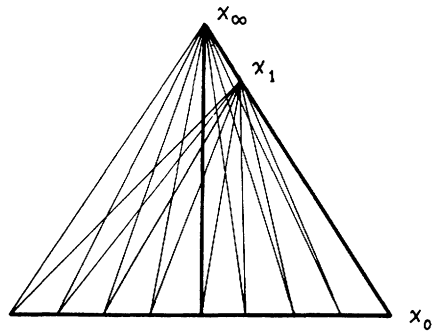

FIGURE 8

Fix an integer $d > 0$ and consider the set Div_d'' of all divisors of degree d on $\mathbf{P}^{n+2}$ such that

$$(5.9) \qquad x_\infty \notin D \quad \text{and} \quad x_1 \notin \bigcup\{tD: 0 < t \leq 1\}.$$

It is clear that many such divisors exist for each d. For each such divisor D we define the continuous 1-parameter family of maps

$$(5.10) \qquad \Psi_{tD}: d \cdot \mathscr{C}_{p+1, d_0}(\mathbf{P}^{n+1}) \longrightarrow \mathscr{C}_{p+1, dd_0}(\mathbf{P}^{n+1})$$

by setting

$$\Psi_{tD}(d \cdot c) = \pi_t(\varphi_{tD}(c))$$

for $0 \leq t \leq 1$. This map has the following basic property.

LEMMA 5.5. *Restricted to the subset $d \cdot \Sigma\!\!\!/(\mathscr{C}_{p, d_0}(\mathbf{P}^n)) \subset d \cdot \mathscr{C}_{p+1, d_0}(\mathbf{P}^{n+1})$ of suspended p-cycles, the map Ψ_{tD} is the identity for all t.*

Proof. Fix any point $x \in \mathbf{P}^n$ and consider the line $\lambda_x = \overline{xx_0} = \Sigma\!\!\!/\{x\} \in \mathscr{C}_{1,1}(\mathbf{P}^{n+1})$. All of our constructions applied to λ_x take place in the projective plane $\mathbf{P}_x^2 = \lambda_x \# x_\infty = x \# (\overline{x_0 x_\infty}) \subset \mathbf{P}^{n+2}$. Note that $\varphi_{tD}(\lambda_x)$ is a curve of degree d in $\mathbf{P}_x^2$. The map π_1, which is projection away from $x_1 \in \mathbf{P}_x^2$, preserves the plane $\mathbf{P}_x^2$ and carries $\varphi_{tD}(\lambda_x)$ back to $d \cdot \lambda_x$. The lemma now follows easily. □

We can think of Ψ_{tD} essentially as a transformation on weighted sets of points. The argument just given shows clearly that this transformation takes the family of subsets of any given line λ_x into itself. Consequently we also have the following.

LEMMA 5.6. *For any subvariety $X \subset \mathbf{P}^n$, the transformations Ψ_{tD} leave the subspace $d \cdot \mathscr{C}_{p+1, d_0}(\Sigma\!\!\!/ X)$ invariant.*

Recall that we have constructed the Friedlander completion $\mathscr{C}_{p+1}(\Sigma\!\!\!\!/\, X)$ using a family of cycles of the form $\{\Sigma\!\!\!\!/\, x_\alpha\}_{\alpha\in A}$ where $x_\alpha \in \mathscr{C}_{p,\alpha}(X)$ for each α. For simplicity we shall henceforth assume that $\Sigma\!\!\!\!/\, x_{\alpha+\beta} = \Sigma\!\!\!\!/\, x_\alpha + \Sigma\!\!\!\!/\, x_\beta$ for all α and β, as in the basic case where $X = \mathbf{P}^n$. The space $\mathscr{C}_{p+1}(\Sigma\!\!\!\!/\, X)$ is then a simple direct limit. The more general case, where $\mathscr{C}_{p+1}(\Sigma\!\!\!\!/\, X)$ is a homotopy limit, is easily worked out once the general argument is understood. Observe now that by Lemma 5.5 we have

$$\Psi_{tD}(d\cdot x_\alpha) = d\cdot x_\alpha \qquad \text{for all } t \text{ and } \alpha. \tag{5.11}$$

This together with Lemma 5.6 shows that the maps Ψ_{tD} carry over to the direct limit, and so we have the following.

PROPOSITION 5.7. *For any subvariety $X \subset \mathbf{P}^n$ and for all $D \in \mathrm{Div}_d''$, the family of transformations Ψ_{tD} induces a continuous family of maps*

$$\Psi_{tD}\colon d\cdot\mathscr{C}_{p+1}(\Sigma\!\!\!\!/\, X) \longrightarrow \mathscr{C}_{p+1}(\Sigma\!\!\!\!/\, X) \tag{5.12}$$

which is the identity (i.e., the inclusion) for $t = 0$. This family has the property that

$$\Psi_{tD}\,|_{\,d\cdot\Sigma\!\!\!\!/\,(\mathscr{C}_p(X))} = \mathrm{Id} \qquad \textit{for all } t. \tag{5.13}$$

The reason for constructing the maps Ψ_{tD} is that they can be used to deform cycles which lie in the hyperplane $\mathbf{P}^n$ to cycles which do not, i.e., to cycles which lie in the subset $\mathscr{T}$. For a single fixed cycle, an appropriately chosen divisor of degree 1 would suffice. For an m-dimensional family of cycles, however, we must use a divisor of sufficiently high degree d (and the family must first be multiplied by d). We shall find an integer d_m depending only on m with the property that such divisors can be found in all degrees $d \ge d_m$.

We shall begin by examining in detail the question of which cycles remain in $\mathbf{P}^n$ and which cycles are carried into $\mathbf{P}^n$ by the family of transformations Ψ_{tD}. We fix a divisor $D \in \mathrm{Div}_d''$ on $\mathbf{P}^{n+2}$ with property (5.9) and associate to this divisor the subset $\alpha(D) \subset \mathbf{P}^{n+1}$ defined by

$$\alpha(D) = \Psi_D^{-1}(\mathbf{P}^n) = \pi_\infty\big(D\cap\pi_1^{-1}(\mathbf{P}^n)\big) \tag{5.14}$$

where we are abusing notation and thinking of divisors as subsets (or reduced divisors) in $\mathbf{P}^{n+2}$.

LEMMA 5.8. *Fix $D \in \mathrm{Div}_d''$ and let $V \subset \mathbf{P}^{n+1}$ be an irreducible algebraic subvariety. Then*

$$\Psi_D(d\cdot V) \subset \mathbf{P}^n \;\Rightarrow\; V \subset \alpha(D).$$

or equivalently

$$V \not\subset \alpha(D) \;\Rightarrow\; \Psi_D(d\cdot V) \in \mathscr{T}.$$

Proof. For $x \in V$ we have

$$\begin{aligned}\Psi_D(d \cdot x) \stackrel{\text{def}}{=} \pi_1\big(D \cap \pi_\infty^{-1}(x)\big) \subset \mathbf{P}^n \Leftrightarrow\ & D \cap \pi_\infty^{-1}(x) \subset \pi_1^{-1}(\mathbf{P}^n)\\ \Rightarrow\ & \pi_1^{-1}(\mathbf{P}^n) \cap D \cap \pi_\infty^{-1}(x) \neq \varnothing\\ \Rightarrow\ & x \in \pi_\infty\big(\pi_1^{-1}(\mathbf{P}^n) \cap D\big). \qquad \square\end{aligned}$$

Definition 5.9. We shall say that $c = \Sigma_\beta n_\beta V_\beta \in \mathscr{C}_{p+1, d_0}(\mathbf{P}^{n+1})$ *has no component in* $\alpha(D)$ if $V_\beta \not\subset \alpha(D)$ for each β (with $n_\beta > 0$). Otherwise we say that c *has some component in* $\alpha(D)$ and write $c \vDash \alpha(D)$.

For any D as above, $x_0 \notin \alpha(D)$ and therefore $\not\Sigma x \not\subset \alpha(D)$ for any cycle $c \in \mathscr{C}_{p,\cdot}(X)$. This applies in particular to the family $\{\not\Sigma x_\gamma\}_{\gamma \in A}$, and therefore the property that a cycle c has no component in $\alpha(D)$ carries over to the direct limit $\mathscr{C}_{p+1}(\not\Sigma X)$.

Lemma 5.8 has the following corollary.

COROLLARY 5.10. *Suppose* $c \in \mathscr{C}_{p+1}(\not\Sigma X)$ *has no component in* $\alpha(tD)$ *for* $0 < t \leq 1$. *Then*

$$\Psi_{tD}(d \cdot c) \in \mathscr{T} \quad \text{for} \quad 0 < t \leq 1. \tag{5.15}$$

This leads us to consider for any cycle $c \in \mathscr{C}_{p+1,\cdot}(\mathbf{P}^{n+1})$ the "bad set"

$$\mathscr{B}(c) = \{D \in \mathrm{Div}_d'' : c \vDash \alpha(D)\}. \tag{5.16}$$

We want to estimate the codimension of $\mathscr{B}(c)$ in Div_d''. To begin we note that if $c = \Sigma_\beta n_\beta V_\beta$, then

$$\mathscr{B}(c) = \bigcup_\beta \mathscr{B}(V_\beta).$$

Therefore, it suffices to estimate the codimension of $\mathscr{B}(V)$ where $V \subset \mathbf{P}^{n+1}$ is irreducible. Under this assumption $V \vDash \alpha(D) \Leftrightarrow V \subset \alpha(D)$. Now set $H = \pi_1^{-1}(\mathbf{P}^n)$ and note that under the isomorphism of hyperplanes

$$F \stackrel{\text{def}}{=} (\pi_\infty|_H)^{-1} \colon \mathbf{P}^{n+1} \xrightarrow{\simeq} H,$$

we have $F(\alpha(D)) = D \cap \pi_1^{-1}(\mathbf{P}^n)$. Therefore,

$$V \subset \alpha(D) \Leftrightarrow F(V) \subset F(\alpha(D)) \Rightarrow F(V) \subset D.$$

Consequently, setting $\tilde{V} = F(V)$, we see that

$$\begin{aligned}\mathscr{B}(V) &= \{D \in \mathrm{Div}_d'' : \tilde{V} \subset D\}\\ &= \mathbf{P}\{\sigma \in H^0(\mathbf{P}^{n+2}; \mathscr{O}(d)) : \sigma|_{\tilde{V}} = 0\},\end{aligned}$$

from which it follows that

$$\operatorname{codim}(\mathscr{B}(V)) = \operatorname{rank}\{H^0(\mathbf{P}^{n+2}; \mathscr{O}(d)) \to H^0(\tilde{V}; \mathscr{O}(d))\}. \tag{5.17}$$

LEMMA 5.11. *For any* $c \in \mathscr{C}_{p+1,\cdot}(\mathbf{P}^{n+1})$, *one has*

$$\operatorname{codim}_{\mathbf{C}}(\mathscr{B}(c)) \geq \binom{p+d+1}{d}.$$

Proof. It suffices to prove this for $c = V$ as above. We pass to a system of affine coordinates $z = (z_1, \dots, z_{n+2})$ for $\mathbf{P}^{n+2}$ which contains regular points of the $(p+1)$-dimensional subvariety $\tilde{V} = F(V)$. Here the space $H^0(\mathbf{P}^{n+2}; \mathscr{O}(d))$ corresponds exactly to the space P_d of all polynomials in z of degree $\leq d$. Let $\tilde{P}_d$ denote the image of P_d under restriction to $\tilde{V}$. Then

$$\dim(\tilde{P}_d) \geq \binom{p+d+1}{d}$$

with equality occurring if and only if $\tilde{V}$ is a linear subspace. To see this, we suppose without loss of generality that linear projection of $\tilde{V} \cap \mathbf{C}^{n+2}$ onto the $(z_1, \dots, z_{p+1})$-coordinate plane is surjective. The polynomials in $(z_1, \dots, z_{p+1})$ then inject into $\tilde{P}_d$. The lemma now follows from (5.17). □

We are now in a position to prove the main results. For a manifold Y, a map $f: Y \to \mathscr{C}_{p+1,\cdot}(\Sigma X)$ will be called *regular* if it is PL with respect to smooth triangulations of Y and $\mathscr{C}_{p+1,\cdot}(\Sigma X)$. Any map is homotopic to a regular one.

THEOREM 5.12. *Let* $f: S^m \to \mathscr{C}_{p+1,\cdot}(\Sigma X)$ *be a regular map. Then for each* $d \in \mathbf{Z}^+$ *satisfying* $2\binom{p+d+1}{d} > m+1$, *there exists a divisor* $D \in \operatorname{Div}_d''$ *so that the homotopy*

$$f_t = \Psi_{tD} \circ (d \cdot f)$$

has the property that $f_0 = d \cdot f$ *and*

$$f_t(S^m) \subset \mathscr{T} \qquad \text{for} \quad 0 < t \leq 1.$$

Proof. Fix any such integer d and consider the subset

$$\mathscr{B}(f) = \bigcup_{\substack{0 < t \leq 1 \\ x \in S^m}} t \cdot \mathscr{B}(f(x))$$

of Div_d'', where $t \cdot \mathscr{B}(c) = \{tD : D \in \mathscr{B}(c)\}$. From Lemma 5.11 we conclude that

$$\operatorname{codim}_{\mathbf{R}} \mathscr{B}(f) \geq 2\binom{p+d+1}{d} - m - 1 > 0.$$

Consequently, there exists a divisor $D \in \operatorname{Div}_d'' - \mathscr{B}(f)$. The theorem now follows from Corollary 5.10 and the definition (5.16) of $\mathscr{B}(c)$. □

THEOREM 5.13. *Let* $f: (D^{m+1}, S^m) \to (\mathscr{C}_{p+1,\cdot}(\Sigma X), \mathscr{T})$ *be a regular map of pairs. Then for each* $d \in \mathbf{Z}^+$ *satisfying* $2\binom{p+d+1}{d} > m+2$, *there exists a* *divisor* $D \in \mathrm{Div}_d''$ *so that the homotopy*

$$f_t = \Psi_{tD} \circ (d \cdot f)$$

has the property that $f_0 = d \cdot f$ *and*

$$f_t(D^{m+1}) \subset \mathcal{T} \qquad \text{for } 0 < t \leq 1.$$

Proof. Apply the arguments given for Theorem 5.12. □

By passing to the completion $(\mathscr{C}_{p+1}(\Sigma\!\!\!/ X), \mathcal{T})$ of the pair $(\mathscr{C}_{p+1,\cdot}(\Sigma\!\!\!/ X), \mathcal{T}.)$, the last two theorems yield the assertions (5.2′) and (5.3′). Hence, Theorem 3.3 and also Theorem 3.1 are proved. □ □

6. Embellishments

The point of this section is to give some applications and generalizations of our main arguments. We begin with the following.

Proof of Theorem 2. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{SP}^d(\mathbf{P}^q) = \mathscr{C}_{0,d}(\mathbf{P}^q) & \xrightarrow{\Sigma\!\!\!/^p} & \mathscr{C}_{p,d}(\mathbf{P}^{p+q}) \\ \downarrow & & \downarrow \\ \mathrm{SP}(\mathbf{P}^q) = \mathscr{C}_0(\mathbf{P}^q) & \xrightarrow{\Sigma\!\!\!/^p} & \mathscr{C}_p(\mathbf{P}^{p+q}). \end{array}$$

From [D] and [DP] or [Mg] we know that the left vertical arrow is a $2d$-connected mapping, and by Theorem 1 the lower horizontal arrow is a homotopy equivalence. It follows immediately that the right vertical arrow has a right homotopy inverse through dimension $2d$. □

Proof of Lemma 2.11. This amounts to an easy version of the arguments given in Sections 4 and 5. Surjectivity of the map $\Sigma\!\!\!/: \pi_0\mathscr{C}_{p,\cdot}(X) \to \pi_0\mathscr{C}_{p+1,\cdot}(\Sigma\!\!\!/ X)$ follows by placing the cycle in general position with respect to the base $X \subset \Sigma\!\!\!/ X$ and then pulling via the linear flow. General position is achieved by a homotopy Ψ_{tD} as in Section 5; however in this case we may assume $\deg D = 1$. This is important since we are dealing here with a monoid, not a group completion.

Injectivity of the map is proved similarly. Any map $\gamma: [0,1] \to \mathscr{C}_{p+1,\cdot}(\Sigma\!\!\!/ X)$ with $\gamma(0), \gamma(1) \in \Sigma\!\!\!/\mathscr{C}_{p,\cdot}(X)$, can be put into general position and pulled by the linear flow to a map $\tilde{\gamma}: [0,1] \to \Sigma\!\!\!/\mathscr{C}_{p,\cdot}(X)$. This process leaves the endpoints fixed. □

Proof of Theorem 3 (*second part*). One applies the arguments of Sections 4 and 5 with $\mathscr{C}_p(X)$ replaced by $\tilde{\mathscr{C}}_p(X)$. There is only one point requiring some care. One must check carefully that the map $\varphi_D(c)$, discussed in Lemma 5.4, is well-defined and continuous in this case. The definition is given as follows. For $c \in \tilde{\mathscr{C}}_p(X)$, write $c = c^+ - c^-$ where c^+ and c^- are positive cycles, and set

$$\varphi_D(c) = \varphi_D(c^+) - \varphi_D(c^-).$$

Note that $\varphi_D(c^+ + c_0) - \varphi_D(c^- + c_0) = \varphi_D(c^+) - \varphi_D(c^-)$ for any positive cycle c_0. Hence our definition of $\varphi_D(c)$ is independent of the choice of c^+, c^-.

To see that φ_D is continuous, we fix a convergent sequence $c_j \to c$ in $\tilde{\mathscr{C}}_p(X)$ and show that $\varphi_D(c_j) \to \varphi_D(c)$. To see this consider first the filtration

$$\cdots \subset \tilde{\mathscr{C}}_{p,d}(X) \subset \tilde{\mathscr{C}}_{p,d+1}(X) \subset \cdots$$

where

$$\tilde{\mathscr{C}}_{p,d}(X) = \delta\big(\mathscr{C}_{p,d}(X) \times \mathscr{C}_{p,d}(X)\big) \tag{6.1}$$

and where

$$\mathscr{C}_{p,d}(X) = \{c \in \mathscr{C}_{p,\cdot}(X) \colon \text{degree } c \le d\}.$$

(Here degree c refers to the degree of c considered as a cycle in $\mathbf{P}^n$.) By Lemma 2.5 there is a d such that $\{c_j\}_{j=1}^{\infty} \cup c \subset \tilde{\mathscr{C}}_{p,d}(X)$. This means that for each j we can write $c_j = c_j^+ - c_j^-$ where the cycles $c_j^{\pm}$ have degree $\le d$. Hence, by passing to a subsequence if necessary, we may assume that there are effective p-cycles c^+ and c^- of degree $\le d$ such that $c_j^+ \to c^+$ and $c_j^- \to c^-$. Clearly, $c = c^+ - c^-$. By Lemma 5.4 we have that

$$\varphi_D(c_j^+) - \varphi_D(c_j^-) \longrightarrow \varphi_D(c^+) - \varphi_D(c^-) = \varphi_D(c),$$

and the continuity of φ_D is proved. □

Proof of Theorem 3.4. In each case it suffices to construct a cross-section on a neighborhood of the identity in the quotient. This is done inductively over steps in the filtration of the quotient given by the images of the sets $\tilde{\mathscr{C}}_{p,d}(X)$ defined in (6.1). The details follow closely those given in [DT2] for the case $p = 0$. We will not reproduce them here. □

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